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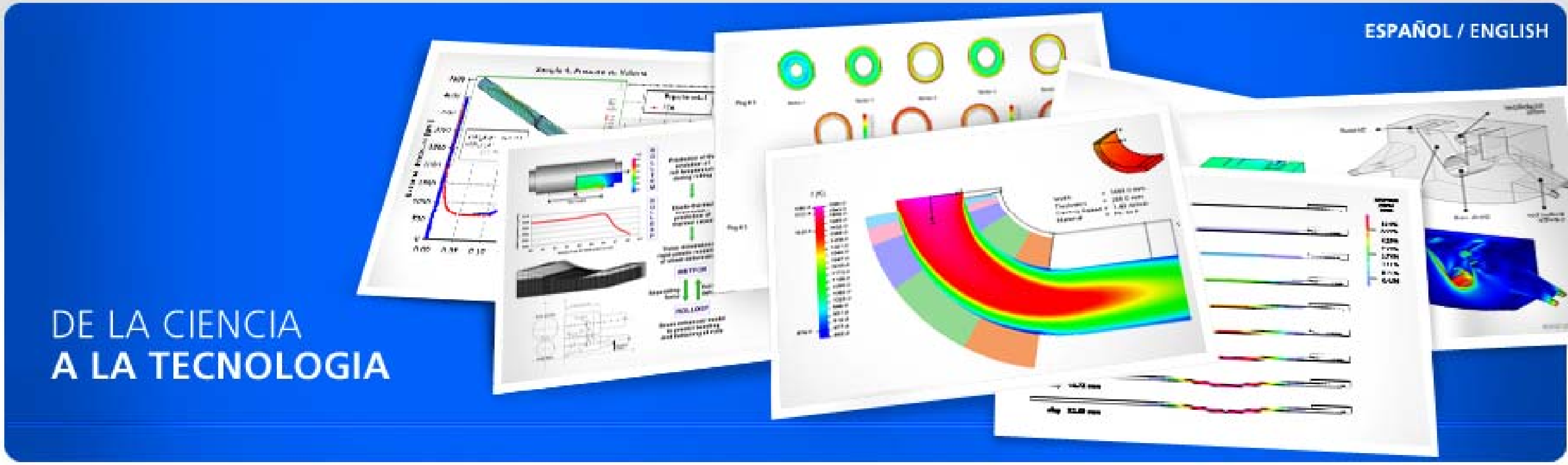

Advanced Topics in Computational Solid Mechanics.

Section 2: Kinematics of the continuous media

Eduardo N. Dvorkin

Stanford University
Mechanical Engineering
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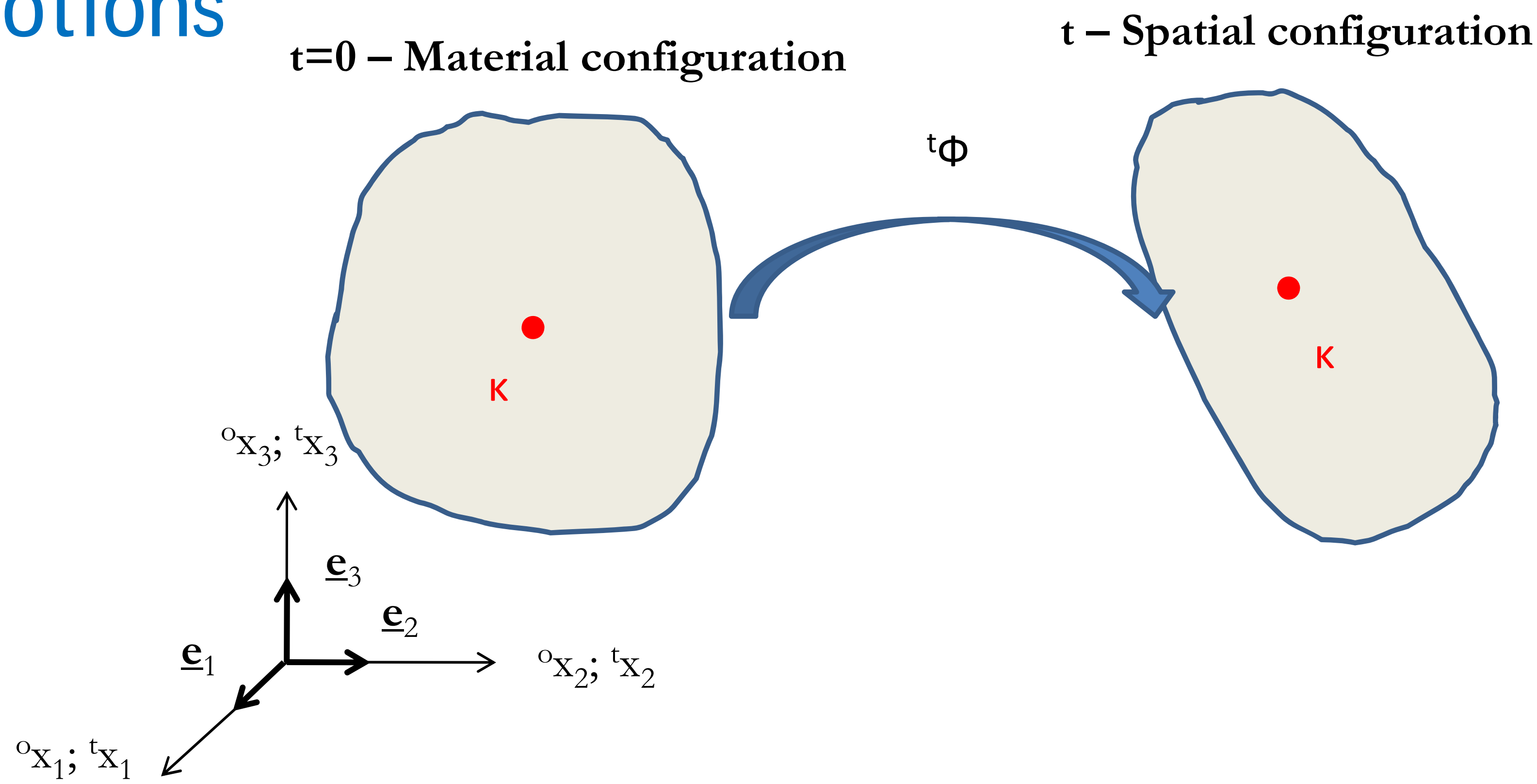
The continuum media and its configuration. Motions



$\underline{e}_\alpha = {}^o\underline{e}_\alpha = {}^t\underline{e}_\alpha$

The continuum media and its configuration

Motions

$$^{t}x_a = {}^{t}x_a(\chi, t) \quad ; \quad \chi = \chi({}^{t}x_a)$$

$$^{t}\tilde{x}_b = {}^{t}\tilde{x}_b({}^{t}x_a) \quad ; \quad {}^{t}x_a = {}^{t}x_a({}^{t}\tilde{x}_b)$$

Spatial configuration

$$^{o}x_A = {}^{o}x_A(\chi) \quad ; \quad \chi = \chi({}^{o}x_A)$$

Material configuration
Reference configuration

Motion:

$$^{t}x_a = {}^{t}\phi_a({}^{o}x_A, t)$$

$$^{t}\tilde{x}_a = {}^{t}\tilde{x}_a({}^{t}x_b) = {}^{t}\tilde{x}_a\left[{}^{t}\phi_b({}^{o}x_A)\right] = {}^{t}\tilde{\phi}_a({}^{o}x_A)$$

Regular motion : no opening of holes and no material interpenetration

$$^{o}x_A = \left[{}^{t}\phi^{-1}\right]_A({}^{t}x_A)$$

The continuum media and its configuration
Motions

$$ {}^{o}\underline{x}(\kappa) = {}^{o}x_{a}(\kappa)\ {}^{o}\underline{e}_{a} \qquad ; \qquad {}^{t}\underline{x}(\kappa) = {}^{t}x_{a}(\kappa)\ {}^{t}\underline{e}_{a} $$

$$ {}^{t}u_{a} = {}^{t}x_{a} - {}^{o}x_{a} $$

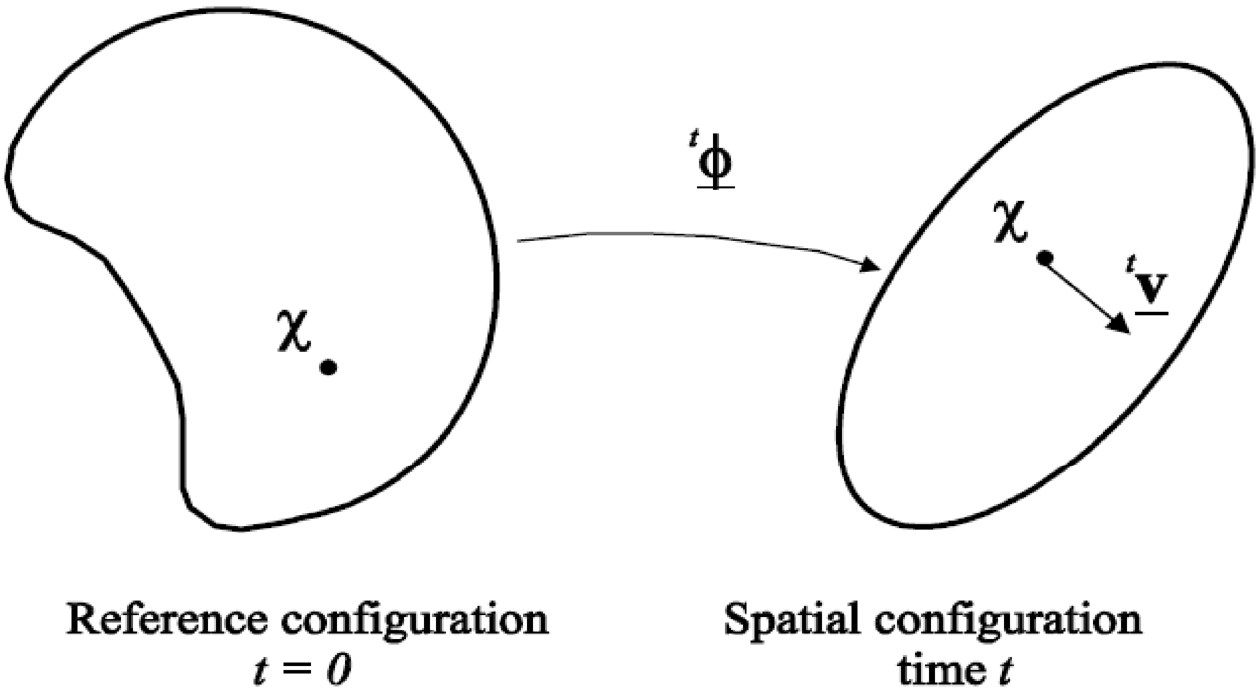
Material velocity:

$$ {}^{t}\underline{\mathbf{v}}(\chi, t) = \frac{\partial\, {}^{t}\underline{\mathbf{x}}(\chi, t)}{\partial t} $$

The continuum media and its configuration
Motions

Lagrangean (material) description of motion	$^{t}\underline{v} = {}^{t}\underline{v}\left({}^{o}x_{A}, t\right)$
Eulerian (spatial) description of motion	$^{t}\underline{v} = {}^{t}\underline{v}\left({}^{t}x_{a}, t\right)$

The continuum media and its configuration
Motions



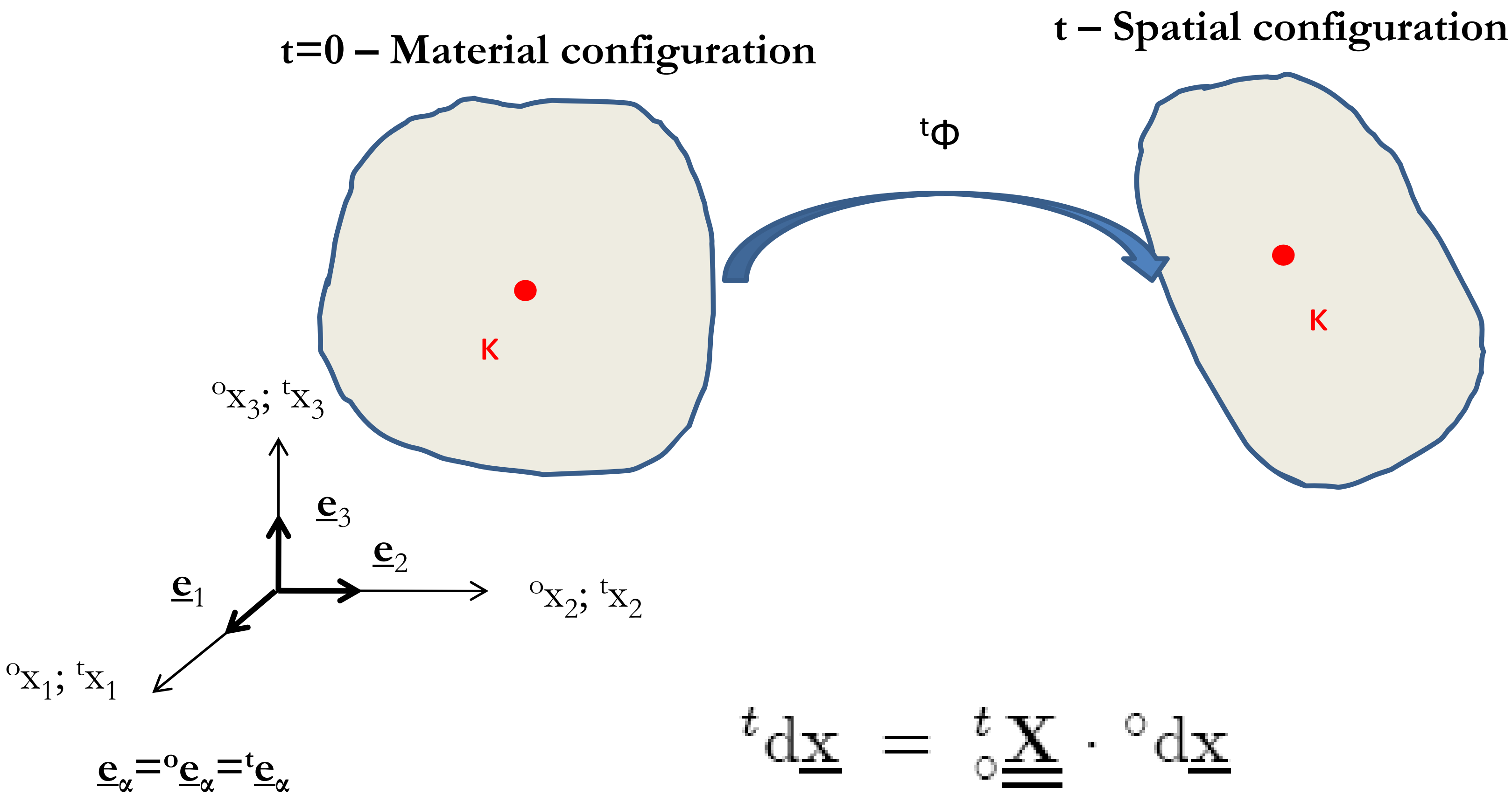
Material velocity of a particle $\quad {}^t\underline{v} = {}^t v_\alpha \, \underline{e}_\alpha$

The continuum media and its configuration Motions

Temporal material derivative: following the particle $$ {}^t\underline{\mathbf{a}} = \frac{D\,{}^t\underline{\mathbf{v}}}{Dt} . $$

Description	*Coordinates*	
Lagrangean (material) description of motion	*Fixed Cartesian*	$${}^t\underline{a} = \frac{\partial\, {}^t v_\alpha}{\partial t}\,\underline{e}_\alpha$$
Eulerian (spatial) description of motion	*Fixed Cartesian*	$${}^t\underline{a} = \left[\frac{\partial\, {}^t v_\alpha}{\partial t} + \frac{\partial\, {}^t v_\alpha}{\partial\, {}^t z_\beta}\, {}^t v_\beta\right]\underline{e}_\alpha$$

The deformation gradient tensor



$$ {}^t d\underline{x} = {}^t_o\underline{\underline{X}} \cdot {}^o d\underline{x} $$

The deformation gradient tensor

$$ {}^{t}_{o}\underline{\underline{X}} = \frac{\partial\, {}^{t}x_{a}}{\partial\, {}^{o}x_{A}}\; {}^{t}\underline{e}_{a}\; {}^{o}\underline{e}_{A} $$

The deformation gradient tensor is a two-point tensor

$$ {}^{\circ}\mathrm{d}\underline{\mathrm{x}} = {}^{t}_{\circ}\underline{\underline{\mathrm{X}}}^{-1} \cdot {}^{t}\mathrm{d}\underline{\mathrm{x}} $$

$$ {}^{t}_{o}\underline{\underline{X}}^{-1} = \frac{\partial\, {}^{o}x_{A}}{\partial\, {}^{t}x_{a}}\; {}^{o}\underline{e}_{A}\; {}^{t}\underline{e}_{a} $$

The deformation gradient tensor

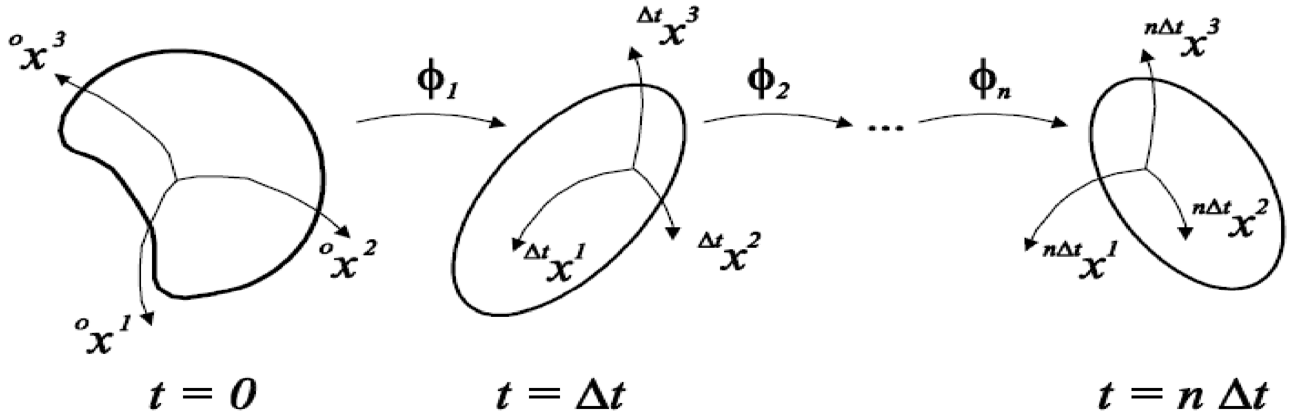


Fig. 2.3. Sequence of motions

$$ {}^{n\Delta t}_{\circ}X_{aP} = \frac{\partial\, {}^{n\Delta t}x_a}{\partial\, {}^{\circ}x_P} = \frac{\partial\, {}^{n\Delta t}x_a}{\partial\, {}^{(n-1)\Delta t}x_b} \cdots\cdots \frac{\partial\, {}^{\Delta t}x_l}{\partial\, {}^{\circ}x_P} $$

Therefore,

$$ {}^{n\Delta t}_{\circ}\underline{\underline{\mathbf{X}}} = {}^{n\Delta t}_{(n-1)\Delta t}\underline{\underline{\mathbf{X}}} \cdot {}^{(n-1)\Delta t}_{(n-2)\Delta t}\underline{\underline{\mathbf{X}}} \cdots {}^{2\Delta t}_{\Delta t}\underline{\underline{\mathbf{X}}} \cdot {}^{\Delta t}_{\circ}\underline{\underline{\mathbf{X}}} \,. $$

The deformation gradient tensor

$$^{t}J(\chi, t) = \frac{^{t}\mathrm{d}V}{^{\circ}\mathrm{d}V}$$

$$^{t}J(\chi, t) = \left|{}^{t}_{\circ}X\right|$$

The right polar decomposition

The Green deformation tensor

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{C}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} $$

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}} \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}} = {}^{\circ}\mathrm{d}\underline{\mathbf{x}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{C}}} \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}} $$

- Symmetric
- Positive-definite

The right polar decomposition

The right stretch tensor

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{U}}} = \left[{}_{\circ}^{t}\underline{\underline{\mathbf{C}}} \right]^{1/2} $$

- Symmetric
- Positive-definite

The right polar decomposition

$$ {}_{\circ}^{t}\underline{\underline{\mathbf{X}}} = {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{U}}} $$

$$ (i) \quad {}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} = {}^{\circ}\underline{\underline{\mathbf{g}}} $$

$$ (ii) \quad {}_{\circ}^{t}\underline{\underline{\mathbf{R}}} \cdot {}_{\circ}^{t}\underline{\underline{\mathbf{R}}}^{T} = {}^{t}\underline{\underline{\mathbf{g}}} $$

${}_{\circ}^{t}\underline{\underline{\mathbf{R}}}$ is orthogonal

The right polar decomposition is unique

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The right polar decomposition
Physical interpretation

$$\text{If} \qquad {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}$$

$${}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2$$

$${}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{C}}} \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \right) \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2$$

The left polar decomposition

The Finger deformation tensor

$$ {}^{t}\underline{\underline{\mathbf{b}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{T} $$

- Symmetric
- Positive-definite

The left polar decomposition

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} $$

The left stretch tensor

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} = \left[{}^{t}\underline{\underline{\mathbf{b}}} \right]^{1/2} $$

The left polar decomposition is unique

The left polar decomposition
Physical interpretation

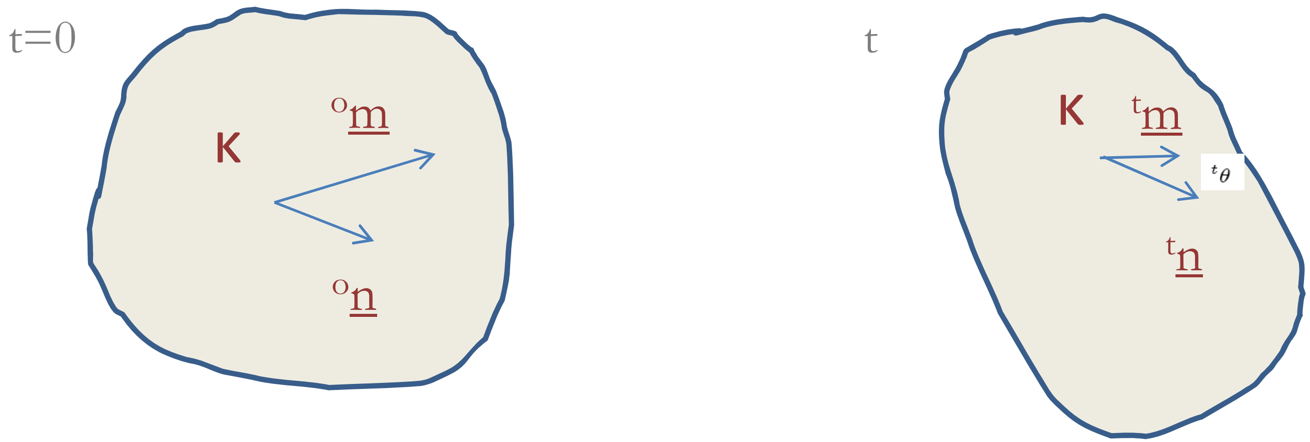
$$ {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 = {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{V}}}^{-1} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{V}}}^{-1} \right) \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 $$

Please notice that:

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{V}}}^{T} = {}^{t}_{\circ}\underline{\underline{\mathbf{R}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{U}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{R}}}^{T} = {}^{t}_{\circ}\underline{\underline{\mathbf{V}}} $$

${}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$ and ${}^{t}_{\circ}\underline{\underline{\mathbf{V}}}$ have the same eigenvalues

The polar decomposition
Physical interpretation



$$\frac{{}^{t}\mathrm{d}S}{{}^{\circ}\mathrm{d}S} = \left[\ [{}^{\circ}n]^{T}\ [{}^{t}_{\circ}C]\ [{}^{\circ}n]\ \right]^{1/2}$$

$${}^{t}\theta = \cos^{-1}\left[\frac{[{}^{\circ}m]^{T}\quad [{}^{t}_{\circ}C]\quad [{}^{\circ}n]}{\left(\frac{{}^{t}\mathrm{d}S}{{}^{\circ}\mathrm{d}S}\right)_{m}\quad \left(\frac{{}^{t}\mathrm{d}S}{{}^{\circ}\mathrm{d}S}\right)_{n}}\right]$$

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Strain measures

Green

$$ {}^{t}d\underline{x}_1 \cdot {}^{t}d\underline{x}_2 = {}^{\circ}d\underline{x}_1 \cdot {}^{t}_{\circ}\underline{\underline{C}} \cdot {}^{\circ}d\underline{x}_2 $$

Finger

$$ {}^{t}d\underline{x}_1 \cdot {}^{t}\underline{\underline{b}}^{-1} \cdot {}^{t}d\underline{x}_2 = {}^{\circ}d\underline{x}_1 \cdot {}^{\circ}d\underline{x}_2 $$

$$ {}^{t}d\underline{x}_1 \cdot {}^{t}d\underline{x}_2 - {}^{\circ}d\underline{x}_1 \cdot {}^{\circ}d\underline{x}_2 = 2\,{}^{\circ}d\underline{x}_1 \cdot \underbrace{\left[\frac{1}{2}\left({}^{t}_{\circ}\underline{\underline{C}} - {}^{\circ}\underline{\underline{g}}\right)\right]}_{\text{Green-Lagrange}\ {}^{t}_{\circ}\underline{\underline{\varepsilon}}} \cdot {}^{\circ}d\underline{x}_2 $$

Green-Lagrange ${}^{t}_{\circ}\underline{\underline{\varepsilon}}$

Strain measures

$$ {}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 - {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_1 \cdot {}^{\circ}\mathrm{d}\underline{\mathbf{x}}_2 = 2\,{}^{t}\mathrm{d}\underline{\mathbf{x}}_1 \cdot \left[\frac{1}{2} \left({}^{t}\underline{\underline{\mathbf{g}}} - {}^{t}\underline{\underline{\mathbf{b}}}^{-1} \right) \right] \cdot {}^{t}\mathrm{d}\underline{\mathbf{x}}_2 $$

Almansi $\quad {}^{t}\underline{\underline{\mathbf{e}}}$

Hencky $\quad {}^{t}_{\circ}\underline{\underline{\mathbf{H}}} = \ln {}^{t}_{\circ}\underline{\underline{\mathbf{U}}}$

Strain rates

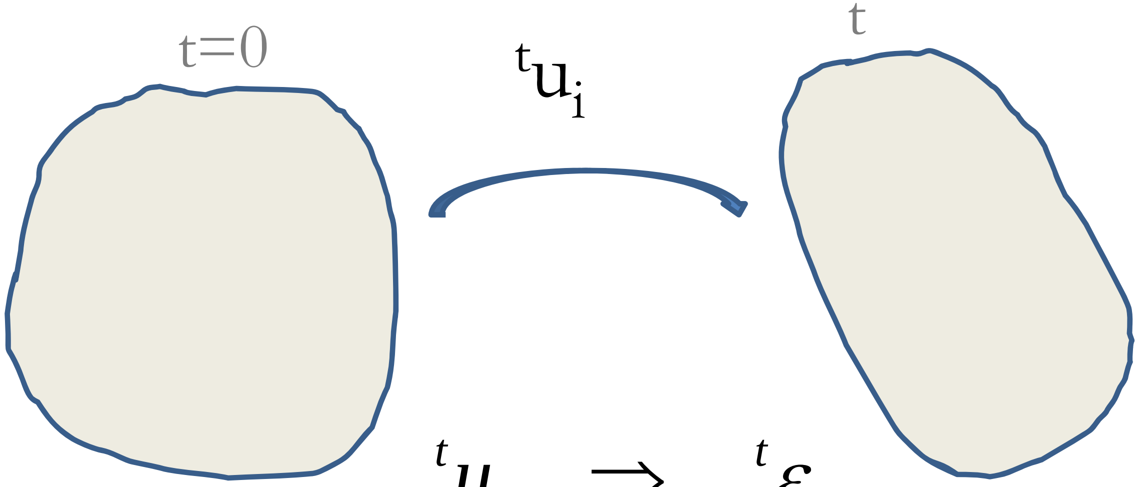
Velocity gradient tensor

$$ {}^{t}\underline{l} = \frac{\partial\, {}^{t}v_i}{\partial\, {}^{t}x_i}\; {}^{t}\underline{e}_i \; {}^{t}\underline{e}_i $$

$$ {}^{t}\underline{\underline{\mathbf{l}}} = {}^{t}\underline{\underline{\mathbf{d}}} + {}^{t}\underline{\underline{\boldsymbol{\omega}}} $$

Strain rate tensor	$ {}^{t}\underline{\underline{\mathbf{d}}} = {}^{t}\underline{\underline{\mathbf{d}}}^{T} = \frac{1}{2}\left({}^{t}\underline{\underline{\mathbf{l}}} + {}^{t}\underline{\underline{\mathbf{l}}}^{T}\right) $
Vorticity tensor	$ {}^{t}\underline{\underline{\boldsymbol{\omega}}} = -\,{}^{t}\underline{\underline{\boldsymbol{\omega}}}^{T} = \frac{1}{2}\left({}^{t}\underline{\underline{\mathbf{l}}} - {}^{t}\underline{\underline{\mathbf{l}}}^{T}\right) $

Compatibility



$$^t u_i \Rightarrow {}^t\varepsilon_{ij}$$

$^t\varepsilon_{ij} \Rightarrow {}^t u_i$ If

$$\frac{\partial^2\ {}^t\varepsilon_{\alpha\beta}}{\partial^\circ z^\gamma\ \partial^\circ z^\delta} + \frac{\partial^2\ {}^t\varepsilon_{\gamma\delta}}{\partial^\circ z^\alpha\ \partial^\circ z^\beta} - \frac{\partial^2\ {}^t\varepsilon_{\alpha\delta}}{\partial^\circ z^\gamma\ \partial^\circ z^\beta} - \frac{\partial^2\ {}^t\varepsilon_{\gamma\beta}}{\partial^\circ z^\alpha\ \partial^\circ z^\delta} = 0$$