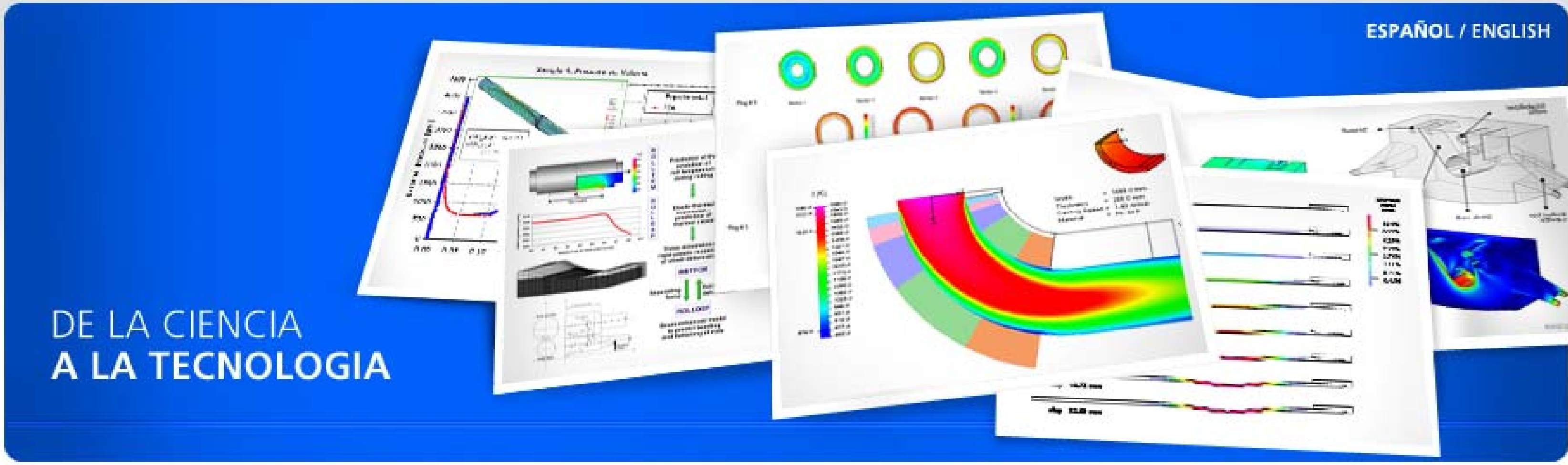
SIM&TEC

Simulación y Tecnología

Simulation and Technology



Advanced Topics in Computational Solid Mechanics.

Section 5: Constitutive relations

Eduardo N. Dvorkin

Stanford University
Mechanical Engineering
Winter Quarter 2010

Constitutive relations: Fundamentals

- For studying the fundamentals that have to be considered for formulating constitutive relations some reference books are: (Truesdell & Noll 1965, Malvern 1969, Marsden & Hughes 1983).
- For studying hyperelasticity: (Ogden 1984).
- For studying plasticity: (Hill 1950, Mendelson 1968, Johnson & Mellor 1973, Lubliner 1990, Simo & Hughes 1998, Kojić & Bathe 2005).
- For studying viscoplasticity: (Perzyna 1966, Kojić & Bathe 2005).
- For studying viscoelasticity: (Pipkin 1972).
- For studying damage mechanics: (Lamaitre & Chaboche 1990).

Constitutive relations: Fundamentals

- Principle of equipresence

- Principle of material-frame indifference

Quoting (Ogden 1984), we can describe classical material objectivity as:
"*An important assumption in continuum mechanics is that two observers in relative motion make equivalent (mathematical and physical) deductions about the macroscopic properties of a material under test. In other words, material properties are unaffected by a superposed rigid motion, and the relation between the stress and the motion has the same form for all observers*".

Constitutive relations: Fundamentals

In this course we add:

- Deterministic relations
- Local action

Elasticity

An *elastic material model* (also called *Cauchy elastic material* (Ogden 1984)) predicts a material behavior independent of the material history and time, that is to say, stresses are univocally determined by strains and vice versa.

$$P_\sigma = \underline{\underline{\mathbf{T}}}^T : \underline{\underline{\dot{\mathbf{E}}}} \,,$$

for an elastic solid,

$$\underline{\underline{\mathbf{T}}} = \underline{\underline{\mathbf{T}}}(\underline{\underline{\mathbf{E}}}) \,,$$

hence,

$$P_\sigma = \underline{\underline{\mathbf{T}}}^T(\underline{\underline{\mathbf{E}}}) : \underline{\underline{\dot{\mathbf{E}}}}$$

Hyperelasticity

The elastic material is hyperelastic if we can write,

$$\dot{U} = \frac{\partial U}{\partial \underline{\underline{\mathbf{E}}}} : \dot{\underline{\underline{\mathbf{E}}}} = \underline{\underline{\mathbf{T}}}^T(\underline{\underline{\mathbf{E}}}) : \dot{\underline{\underline{\mathbf{E}}}}$$

Hence,

$$\underline{\underline{\mathbf{T}}}^T(\underline{\underline{\mathbf{E}}}) = \frac{\partial U}{\partial \underline{\underline{\mathbf{E}}}}$$

Hyperelasticity

Example 5.4. ◀◀◀◀◀

For a hyperelastic material,

$$T_{ij} = \frac{\partial U}{\partial E_{ij}} \quad ; \quad T_{kl} = \frac{\partial U}{\partial E_{kl}}$$

and since

$$\frac{\partial^2 U}{\partial E_{ij}\, \partial E_{kl}} = \frac{\partial^2 U}{\partial E_{kl}\, \partial E_{ij}} ,$$

we must have

$$\frac{\partial T_{ij}}{\partial E_{kl}} = \frac{\partial T_{kl}}{\partial E_{ij}} .$$

◀◀◀◀◀

Hipoelasticity

The *inelastic mechanical behavior* of some materials can be described with equations of the form,

$$d\underline{\underline{\mathbf{T}}} = \underline{\underline{\underline{\underline{\mathbf{C}}}}}(\underline{\underline{\mathbf{T}}}\,,\,\underline{\underline{\mathbf{E}}}) : d\underline{\underline{\mathbf{E}}}\,, \tag{5.2}$$

which are the *hypoelastic material models*.

Hyperelastic material models

Elastic energy in the spatial configuration per unit volume of the reference configuration:

$$d\,{}_{o}^{t}U = {}_{o}^{t}S_{IJ}\,d\,{}_{o}^{t}\varepsilon_{IJ}$$

Elastic energy in the spatial configuration per unit mass:

$$d\,{}^{t}U = \frac{1}{{}^{o}\rho}\,{}_{o}^{t}S_{IJ}\,d\,{}_{o}^{t}\varepsilon_{IJ}$$

Hyperelastic material models

$$ {}_{o}^{t}S_{IJ} = {}^{o}\rho \frac{\partial\ {}^{t}U}{\partial\ {}_{o}^{t}\varepsilon_{IJ}} $$

$$ {}_{o}^{t}S_{IJ} = 2\ {}^{o}\rho \frac{\partial\ {}^{t}U}{\partial\ {}_{o}^{t}C_{IJ}} $$

Also,

$$ {}_{o}^{t}P_{aB} = {}^{o}\rho \frac{\partial\ {}^{t}U}{\partial\ {}_{o}^{t}X_{aB}} $$

A simple hyperelastic model

$$ {}_{\circ}^{t}\mathrm{U} = {}_{\circ}^{t}\mathrm{U}({}_{\circ}^{t}\varepsilon_{IJ}) $$

The simplest possible relation:

$$ {}_{\circ}^{t}\mathrm{U} = {}_{\circ}^{t}A_{\circ} + {}_{\circ}^{t}B_{IJ} \; {}_{\circ}^{t}\varepsilon_{IJ} + \frac{1}{2} \, {}_{\circ}^{t}\hat{C}_{IJKL} \; {}_{\circ}^{t}\varepsilon_{IJ} \; {}_{\circ}^{t}\varepsilon_{KL} $$

A simple hyperelastic model

Use:

$$ {}_{\circ}^{t}A_{\circ} = 0 $$

$$ {}_{o}^{t}S_{IJ} = {}_{o}^{t}B_{IJ} + \frac{1}{2}\left({}_{o}^{t}\hat{C}_{IJKL} + {}_{o}^{t}\hat{C}_{KLIJ}\right){}_{o}^{t}\varepsilon_{KL} $$

Since we are not considering initial stresses,

$$ {}_{o}^{t}B_{IJ} = 0 $$

SIM&TEC Simulación y Tecnología Simulation and Technology

A simple hyperelastic model

Hence, the simplest hyperelastic model is,

$$ {}_o^tU = \frac{1}{2}\ {}_o^t\hat{C}_{IJKL}\, {}_o^t\varepsilon_{IJ}\ {}_o^t\varepsilon_{KL} $$

$$ {}_o^tS_{IJ} = {}_o^tC_{IJKL}\ {}_o^t\varepsilon_{KL} $$

where,

$$ {}_o^tC_{IJKL} = \frac{1}{2}\left({}_o^t\hat{C}_{IJKL} + {}_o^t\hat{C}_{KLIJ}\right) $$

Langrangean and Eulerian models

From

$$ {}_{o}^{t}S_{IJ} = {}_{o}^{t}C_{IJKL}\,{}_{o}^{t}\varepsilon_{KL} $$

After some algebra,

$$ {}^{t}\tau_{ij} = {}^{t}c_{ijkl}\;{}^{t}e_{kl} $$

with,

$$ {}^{t}c_{ijkl} = {}_{o}^{t}C_{IJKL}\,{}_{o}^{t}X_{iI}\,{}_{o}^{t}X_{jJ}\,{}_{o}^{t}X_{kK}\,{}_{o}^{t}X_{lL} $$

SIM&TEC
Simulación y Tecnología
Simulation and Technology

Langrangean and Eulerian models

$$ {}^{t}c_{ijkl} = {}^{t}_{o}C_{IJKL}\ {}^{t}_{o}X_{iI}\ {}^{t}_{o}X_{jJ}\ {}^{t}_{o}X_{kK}\ {}^{t}_{o}X_{lL} $$

Note that the obtained spatial elasticity tensor has components that are a function of the deformation (not constant).

Lagrangean Model	Eulerian Model
Linear	Nonlinear
Nonlinear	Linear

Symmetries

Cause	Consequence
${}^{t}_{o}S_{IJ} = {}^{t}_{o}S_{JI}$	${}^{t}_{o}C_{IJKL} = {}^{t}_{o}C_{JIKL}$
${}^{t}_{o}\varepsilon_{KL} = {}^{t}_{o}\varepsilon_{LK}$	${}^{t}_{o}C_{IJKL} = {}^{t}_{o}C_{IJLK}$
$\frac{\partial\, {}^{t}_{o}S_{IJ}}{\partial\, {}^{t}_{o}\varepsilon_{KL}} = \frac{\partial\, {}^{t}_{o}S_{KL}}{\partial\, {}^{t}_{o}\varepsilon_{IJ}}$	${}^{t}_{o}C_{IJKL} = {}^{t}_{o}C_{KLIJ}$

Without introducing any symmetry inherent to a particular material model, for the description of the most general linear hyperelastic material model we have to use 21 material constants.

Material Symmetries

1 symmetry plane	From 21 to 13 constants
3 symmetry planes (orthotropic)	From 13 to 9 constants
Isotropic	2 constants

SIM&TEC
Simulación y Tecnología
Simulation and Technology

Orthotropic Elasticity

$$\begin{bmatrix} {}_{o}^{t}S_{11} \\ {}_{o}^{t}S_{22} \\ {}_{o}^{t}S_{33} \\ {}_{o}^{t}S_{12} \\ {}_{o}^{t}S_{23} \\ {}_{o}^{t}S_{31} \end{bmatrix} = \begin{bmatrix} {}_{o}^{t}C_{1111} & {}_{o}^{t}C_{1122} & {}_{o}^{t}C_{1133} & 0 & 0 & 0 \\ & {}_{o}^{t}C_{2222} & {}_{o}^{t}C_{2233} & 0 & 0 & 0 \\ & & {}_{o}^{t}C_{3333} & 0 & 0 & 0 \\ & & & {}_{o}^{t}C_{1212} & 0 & 0 \\ & & & & {}_{o}^{t}C_{2323} & 0 \\ SYM & & & & & {}_{o}^{t}C_{3131} \end{bmatrix} \begin{bmatrix} {}_{o}^{t}\varepsilon_{11} \\ {}_{o}^{t}\varepsilon_{22} \\ {}_{o}^{t}\varepsilon_{33} \\ 2\,{}_{o}^{t}\varepsilon_{12} \\ 2\,{}_{o}^{t}\varepsilon_{23} \\ 2\,{}_{o}^{t}\varepsilon_{31} \end{bmatrix}$$

Isotropic Elasticity

$$
\begin{bmatrix} {}_o^tS_{11} \\ {}_o^tS_{22} \\ {}_o^tS_{33} \\ {}_o^tS_{12} \\ {}_o^tS_{23} \\ {}_o^tS_{31} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{(1-\nu)} & \frac{\nu}{(1-\nu)} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{(1-\nu)} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ SYM & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} {}_o^t\varepsilon_{11} \\ {}_o^t\varepsilon_{22} \\ {}_o^t\varepsilon_{33} \\ 2\,{}_o^t\varepsilon_{12} \\ 2\,{}_o^t\varepsilon_{23} \\ 2\,{}_o^t\varepsilon_{31} \end{bmatrix}
$$

$$
{}_\circ^tS_{\alpha\beta} = K\,{}_\circ^t\varepsilon_V\,\delta_{\alpha\beta} + 2\,G\,{}_\circ^t\varepsilon'_{\alpha\beta}
$$

$$
{}_\circ^t\mathrm{U} = \frac{1}{2}\,K\,{}_\circ^t\varepsilon_V{}^2 + G\,{}_\circ^t\varepsilon'_{\alpha\beta}\,{}_\circ^t\varepsilon'_{\alpha\beta}
$$

$$
E \geq 0
$$

$$
-1 \leq \nu \leq 0.5
$$

Odgen Hyperelastic Models

For an isotropic material we can write,

$$ {}_{\circ}^{t}\mathrm{U} = {}_{\circ}^{t}\mathrm{U}(\lambda_1, \lambda_2, \lambda_3) \tag{5.25} $$

where the λ_i are the eigenvalues of the second order tensor, ${}_{\circ}^{t}\underline{\underline{\mathbf{U}}}$, that is to say, the principal stretches defined in Eq. (2.58e).

Since

$$ \lambda_i = \lambda_i(I_1^C, I_2^C, I_3^C) \tag{5.26} $$

where the values (I_1^C, I_2^C, I_3^C) are the invariants of ${}_{\circ}^{t}\underline{\underline{\mathbf{C}}}$ defined in Eqs. (2.59b-2.59d); we can write

$$ {}_{\circ}^{t}\mathrm{U} = {}_{\circ}^{t}\mathrm{U}(I_1^C, I_2^C, I_3^C) \tag{5.27a} $$

$$ I_1^C = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 \tag{5.27b} $$

$$ I_2^C = (\lambda_2)^2 (\lambda_3)^2 + (\lambda_3)^2 (\lambda_1)^2 + (\lambda_1)^2 (\lambda_2)^2 \tag{5.27c} $$

$$ I_3^C = (\lambda_1)^2 (\lambda_2)^2 (\lambda_3)^2 \ . \tag{5.27d} $$

Odgen Hyperelastic Models

$$ {}_{\circ}^{t}\mathrm{U}(\lambda_1, \lambda_2, \lambda_3) = \sum_{p,q,r=0}^{\infty} C_{pqr} \left(I_1^C - 3\right)^p \; \left(I_2^C - 3\right)^q \; \left(I_3^C - 1\right)^r $$

$$ {}^{t}\mathrm{d}V = \lambda_1 \lambda_2 \lambda_3 \; {}^{\circ}\mathrm{d}V $$

$$ {}^{t}J \; = \frac{{}^{t}\mathrm{d}V}{{}^{\circ}\mathrm{d}V} = \sqrt{I_3^C} \; . $$

Odgen Hyperelastic Models

Simplification,

$$
{}^{t}_{\circ}\mathrm{U}(\lambda_1, \lambda_2, \lambda_3) = \sum_{p,q=0}^{\infty} C_{pq0} \left(I_1^C - 3\right)^p \ \left(I_2^C - 3\right)^q
$$

$$
+ \sum_{r=1}^{\infty} C_{00r} \left(I_3^C - 1\right)^r \ .
$$

Elastoplastic material model

- For loads below a certain *limit loading condition*, established via a *yield criterion*, the material behavior is elastic and can be described using the hyperelastic relations that we have discussed above.
- When the limit loading condition is reached there is an onset of *permanent or plastic deformations*.
- The plastic deformations produce an evolution of the yield condition that is described via a *hardening law*.
- When the limit loading condition is reached and then an *unloading* takes place, elastic deformations are developed.
- The material behavior is *rate independent*, that is to say, the material behavior is not a function of the loading or deformation rate.
- The material is stable: we have to spend mechanical work in order to deform it.

1D: Tensile test of a steel sample

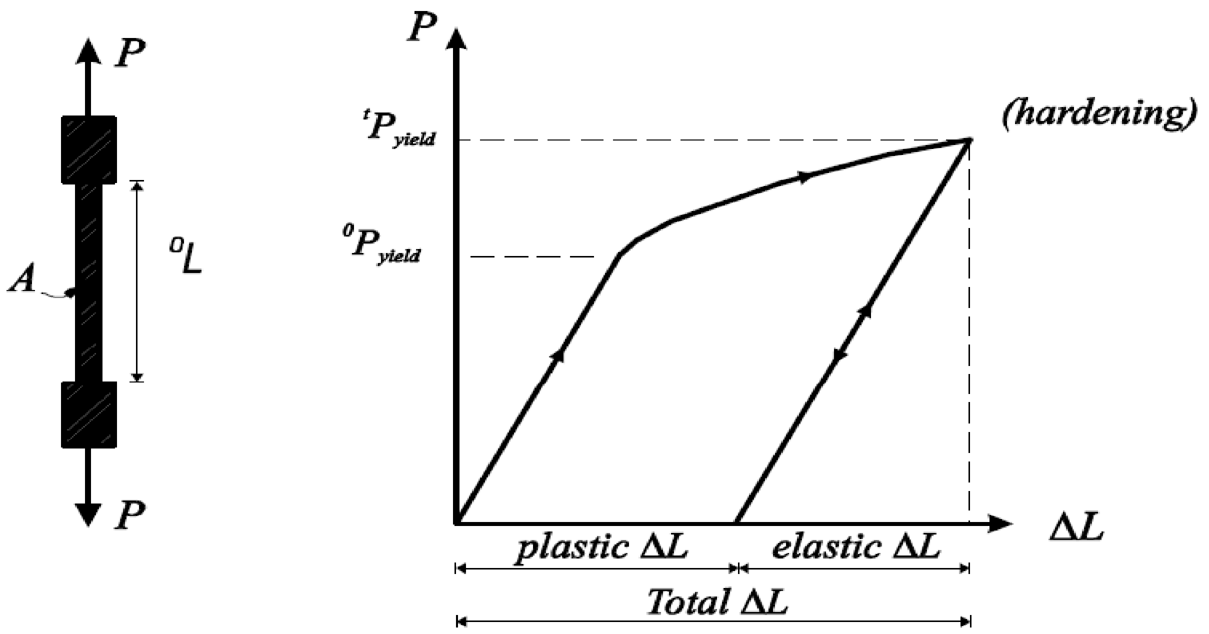


Fig. 5.1. Tensile test of a steel sample

$$^{t}\varepsilon = {}^{t}\varepsilon^{E} + {}^{t}\varepsilon^{P}$$

1D: Tensile test of a steel sample

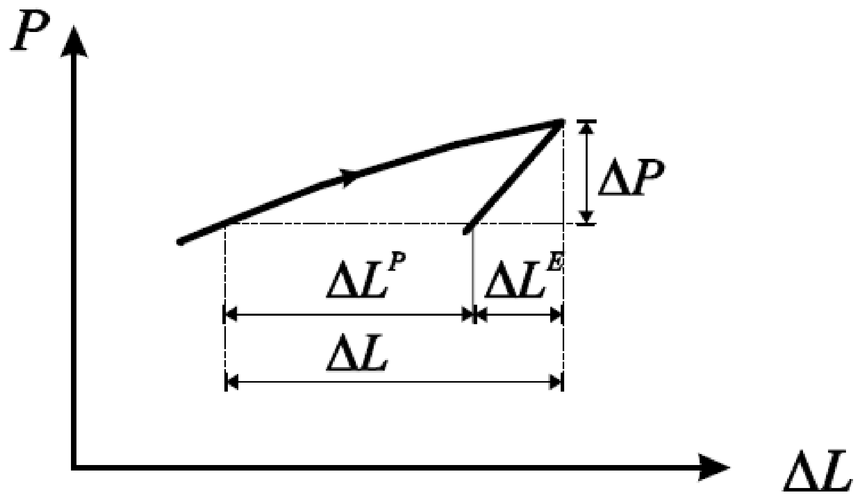


Fig. 5.2. Plastic loading (zoom)

$$\triangle\varepsilon = \triangle\varepsilon^E + \triangle\varepsilon^P$$

1D: Tensile test of a steel sample

$$ {}^{t}\dot{\varepsilon} = {}^{t}\dot{\varepsilon}^{E} + {}^{t}\dot{\varepsilon}^{P} $$

For 3D problems we will generalize the above as,

$$ {}^{t}\underline{\underline{\mathbf{d}}} = {}^{t}\underline{\underline{\mathbf{d}}}^{E} + {}^{t}\underline{\underline{\mathbf{d}}}^{P} $$

1D: Tensile test of a steel sample

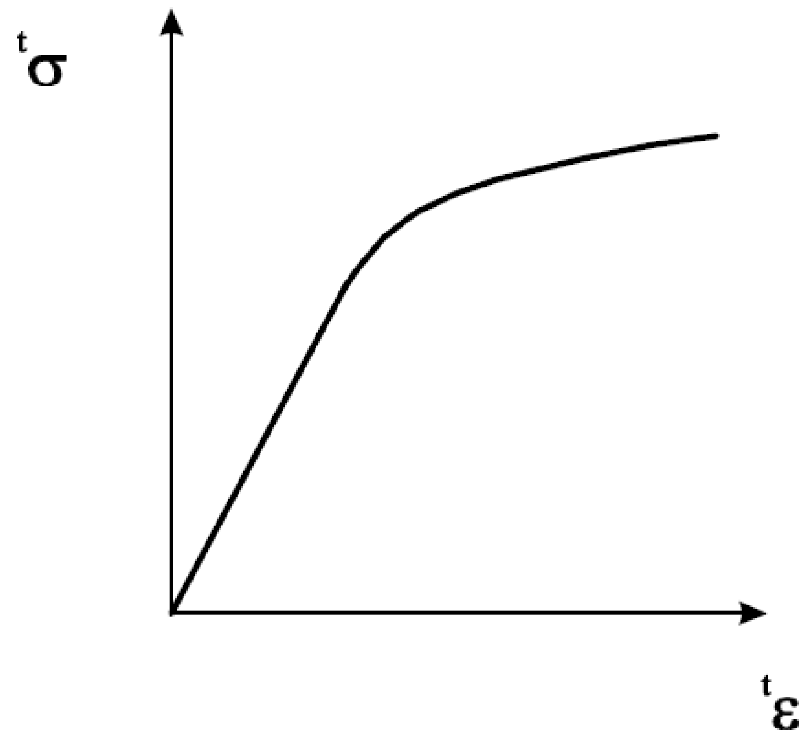


Fig. 5.3. σ-ε for a steel sample

1D: No softening behavior

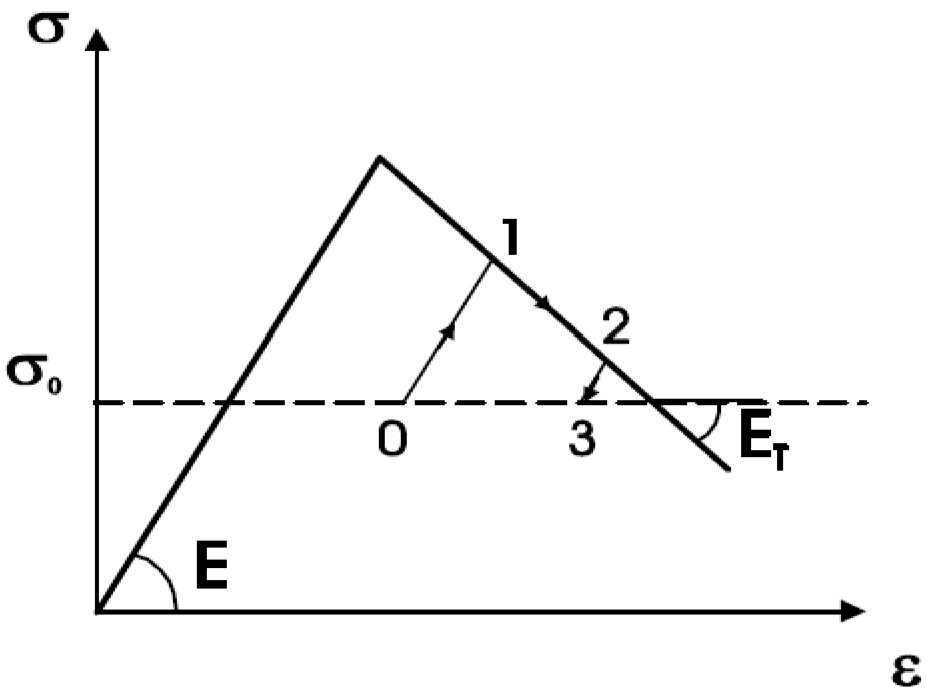


Fig. 5.4. Softening in the $\sigma - \varepsilon$ relation

1D: No softening behavior

In the 1-2 softening path,

$$\triangle\sigma \ \triangle\varepsilon^P \ < \ 0$$

We will now show that the above stress/strain relation is incompatible with the requirement of material stability.

$$W_{e.a.} = \int_0^1 \sigma \ \dot{\varepsilon} \ \mathrm{d}t \ + \int_1^2 \sigma \ \dot{\varepsilon} \ \mathrm{d}t \ + \int_2^3 \sigma \ \dot{\varepsilon} \ \mathrm{d}t$$
$$- \int_0^1 \sigma_\circ \ \dot{\varepsilon} \ \mathrm{d}t - \int_1^2 \sigma_\circ \ \dot{\varepsilon} \ \mathrm{d}t - \int_2^3 \sigma_\circ \ \dot{\varepsilon} \ \mathrm{d}t \ . \qquad (5.43)$$

For a stable material it must be

$$W_{e.a.} \geqslant 0 \ . \qquad (5.44)$$

1D: No softening behavior

Taking into account that in the paths $0-1$ and $2-3$ we are inside the elastic range, we can write

$$W_{e.a.} = \int_0^1 (\sigma - \sigma_\circ)\ \dot{\varepsilon}^E\ \mathrm{d}t\ + \int_1^2 (\sigma - \sigma_\circ)\ \dot{\varepsilon}^E\ \mathrm{d}t\ + \int_2^3 (\sigma - \sigma_\circ)\ \dot{\varepsilon}^E\ \mathrm{d}t\ + \int_1^2 (\sigma - \sigma_\circ)\ \dot{\varepsilon}^P\ \mathrm{d}t\ . \tag{5.45}$$

The first three integrals on the r.h.s. of the above equation add up to zero because they correspond to a loading-unloading elastic cycle and there is neither energy dissipated nor generated in that cycle.

$$W_{e.a.} = \int_1^2 (\sigma - \sigma_\circ)\ \dot{\varepsilon}^P\ \mathrm{d}t\ . \tag{5.46}$$

1D: No softening behavior

When "1" and "2" are infinitesimaly close,

$$d^2 W_{e.a.} = d\sigma \; d\varepsilon^P \;\; \leqslant 0 \; .$$

The points on the softening branch are in unstable equilibrium.

The general formulation

For the general formulation of an elastoplastic material model we need the following three ingredients:

- A *yield surface* that in the 3D stress space describes the locus of the points where the plastic behavior is initiated.
- A *flow rule* that describes the evolution of the plastic deformations.
- A *hardening law* that describes the evolution of the yield surface during the plastic deformation process.

The yield surface

$$ {}^{t}f\left({}^{t}\underline{\underline{\sigma}} \, , \; {}^{t}q_i \;\; i = 1, n \right) = 0 $$

Elastic range:

$$ {}^{t}f < 0 $$

Plastic range:

$$ {}^{t}f = 0 $$

The yield surface for metals

von Mises (J_2)

In his experimental work, developed in the 1950s, Bridgman found that for metals, it can be assumed that the yield function is not affected by the confining hydrostatic pressure - at least for not very extreme hydrostatic pressures (Hill 1950 and Johnson & Mellor 1973).

Decomposition of the stress tensor into a hydrostatic term and a deviatoric one

$$\sigma_{ij} = s_{ij} + p\delta_{ij}$$

$$p = \frac{1}{3}\sigma_{lm}\delta_{lm}$$

The yield surface for metals

The deviatoric stress tensor is traceless

$$ {}^{t}s_{pq}\,\delta_{pq} = 0 $$

J_2: second invariant of ${}^{t}s_{pq}$

J_3 : third invariant of ${}^{t}s_{pq}$

BRIDGMAN ➡ $$ {}^{t}f\left({}^{t}J_2 \,,\, {}^{t}J_3 \,,\, {}^{t}q_i \;\; i = 1, n \right) = 0 $$

SIM&TEC
Simulación y Tecnología
Simulation and Technology

The yield surface for metals

Experimental results indicate that when the yield surface is intersected in the stress space with a plane that forms equal angles with the three principal stress axes, a good approximation for the obtained curve is a circle.

Therefore, von Mises proposed

$$ {}^{t}f\left({}^{t}J_2 \,,\ \ {}^{t}q_i \ i = 1, n \right) = 0 \ . \tag{5.54} $$

Hence, the von Mises yield function is also known as the ${}^{t}J_2$ *-yield function* (Simo & Hughes 1998).

The yield surface for metals

More specifically the actual form of Eq. (5.54) can be written as,

$$ {}^{t}f = \frac{1}{2}\left({}^{t}\underline{\underline{s}} - {}^{t}\underline{\underline{\alpha}} \right) : \left({}^{t}\underline{\underline{s}} - {}^{t}\underline{\underline{\alpha}} \right) - \frac{\left({}^{t}\sigma_y\right)^2}{3} = 0 $$

or,

$$ {}^{t}f = \left[\frac{3}{2}\left({}^{t}\underline{\underline{s}} - {}^{t}\underline{\underline{\alpha}} \right) : \left({}^{t}\underline{\underline{s}} - {}^{t}\underline{\underline{\alpha}} \right)\right]^{1/2} - {}^{t}\sigma_y = 0 \ . $$

The yield surface for metals

Internal variables:

${}^{t}\sigma_y$: uniaxial yield stress at the t-configuration; that is to say corresponding to a given plastic deformation. The evolution of ${}^{t}\sigma_y$ is going to be described by the hardening law.

${}^{t}\underline{\underline{\alpha}}$: back-stress tensor at the t-configuration. In many metals subjected to cyclic loading it is experimentally observed that the center of the yield surface experiences a motion in the direction of the plastic flow; the back-stress tensor describes this behavior (Mc Clintock & Argon 1966). The evolution of ${}^{t}\underline{\underline{\alpha}}$ is going to be described by the hardening law. We will show that ${}^{t}\underline{\underline{\alpha}}$ is a traceless tensor.

The yield surface for metals

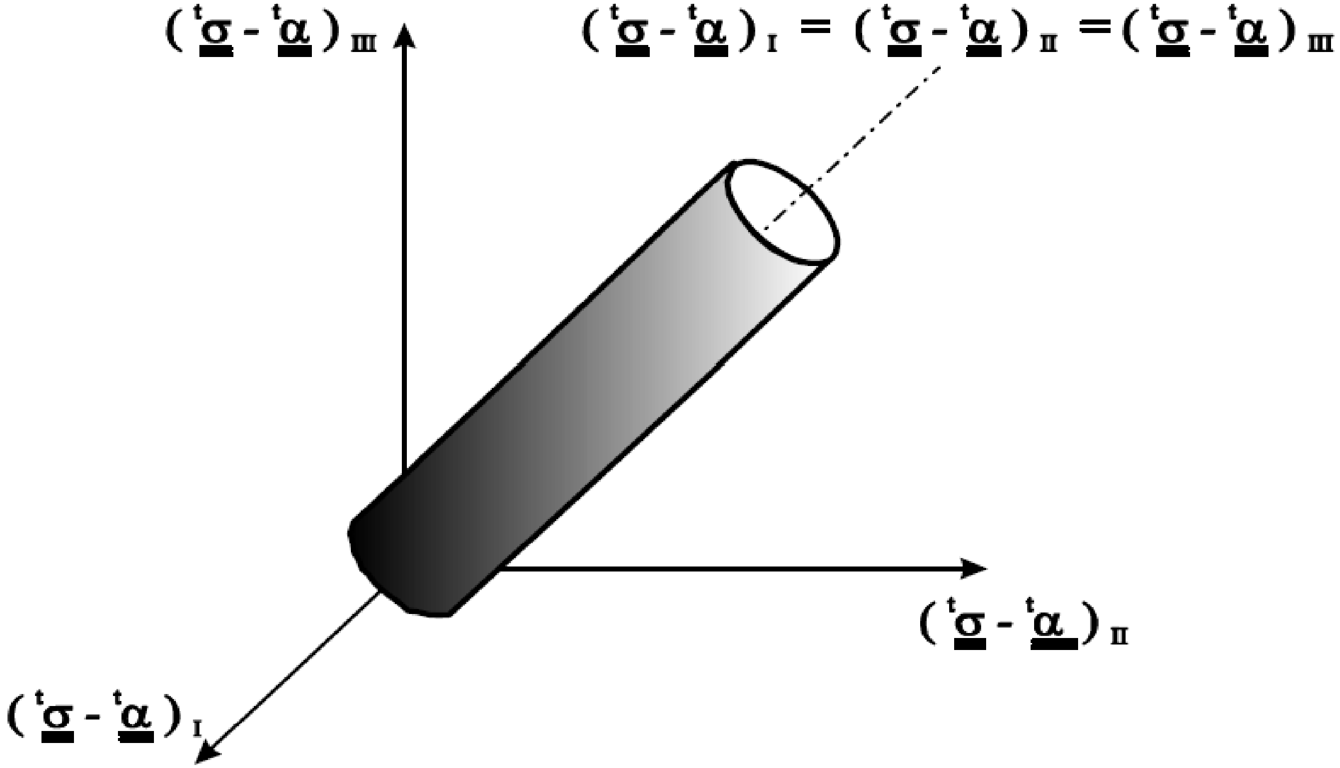


Fig. 5.5. Von Mises yield surface

The flow rule

Plastic dissipation:

$$ {}^{t}D = {}^{t}\sigma_{ij} \; {}^{t}d_{ij}^{P} $$

If ${}^{t}d^{P}_{ij} \neq 0$, due to the Second Law of Thermodynamics: $\quad {}^{t}D > 0$

For many materials, such as metals, the plastic flow is developed so as to maximize the plastic dissipation.

In mathematical form we can say that for defining the plastic loading we seek for the maximum of ${}^{t}\mathcal{D}$ under the constraint

$$ {}^{t}f = 0 \; . \tag{5.58a} $$

We define (Luenberger 1984),

$$ {}^{t}\mathcal{D}^{\star} = {}^{t}\mathcal{D} - {}^{t}\dot{\lambda}\, {}^{t}f \tag{5.58b} $$

where ${}^{t}\dot{\lambda}$ is a Lagrange multiplier used to enforce the constraint in Eq. (5.58a).

The flow rule

$$\frac{\partial \ {}^{t}D^{*}}{\partial \ {}^{t}\sigma_{ij}} = 0$$

$$\frac{\partial \ {}^{t}D^{*}}{\partial \ {}^{t}\dot{\lambda}} = 0$$

Hence,

$${}^{t}d_{ij}^{P} = \ {}^{t}\dot{\lambda} \frac{\partial \ {}^{t}f}{\partial \ {}^{t}\sigma_{ij}}$$

Associated plastic flow

The flow rule

$$
\begin{aligned}
{}^{t}f < 0 \implies {}^{t}\dot{\lambda} = 0 \; (elastic\ behavior) \; , \\
{}^{t}f = 0 \implies {}^{t}\dot{\lambda} > 0 \; (plastic\ loading) \; .
\end{aligned}
$$

$$
\begin{aligned}
{}^{t}\dot{\lambda}\,{}^{t}f = 0 \; , \\
{}^{t}\dot{\lambda} \geqslant 0 \; , \\
{}^{t}f \leqslant 0 \; .
\end{aligned}
$$

The plastic flow of metals is incompressible

Example 5.10. ◀◀◀◀◀

For a material model developed using the von Mises yield criterion in Eq. (5.55), we can write in a Cartesian coordinate system:

$$\frac{\partial\, {}^t f}{\partial\, {}^t\sigma_{\alpha\beta}} = \frac{\partial\, {}^t f}{\partial\, {}^t s_{\gamma\delta}} \frac{\partial\, {}^t s_{\gamma\delta}}{\partial\, {}^t\sigma_{\alpha\beta}} = \left({}^t s_{\alpha\beta} - {}^t\alpha_{\alpha\beta} \right) \quad .$$

Hence, in the case of associated plastic flow

$${}^t d^P_{\alpha\beta} = {}^t\dot{\lambda} \left({}^t s_{\alpha\beta} - {}^t\alpha_{\alpha\beta} \right) \quad .$$

Since ${}^t\underline{\underline{s}}$ and ${}^t\underline{\underline{\alpha}}$ are traceless tensors (see Eqs. (5.78) and (5.52)), it is obvious that,

$${}^t d^P_{\alpha\alpha} = 0 \ ,$$

which is the condition of incompressible plastic flow (see Example 4.4).

The above is, of course, a direct consequence of the fact that due to Bridgman experimental observations the yield function does not include the trace of ${}^t\underline{\underline{\sigma}}$. ◀◀◀◀◀

Plastic flow of metals

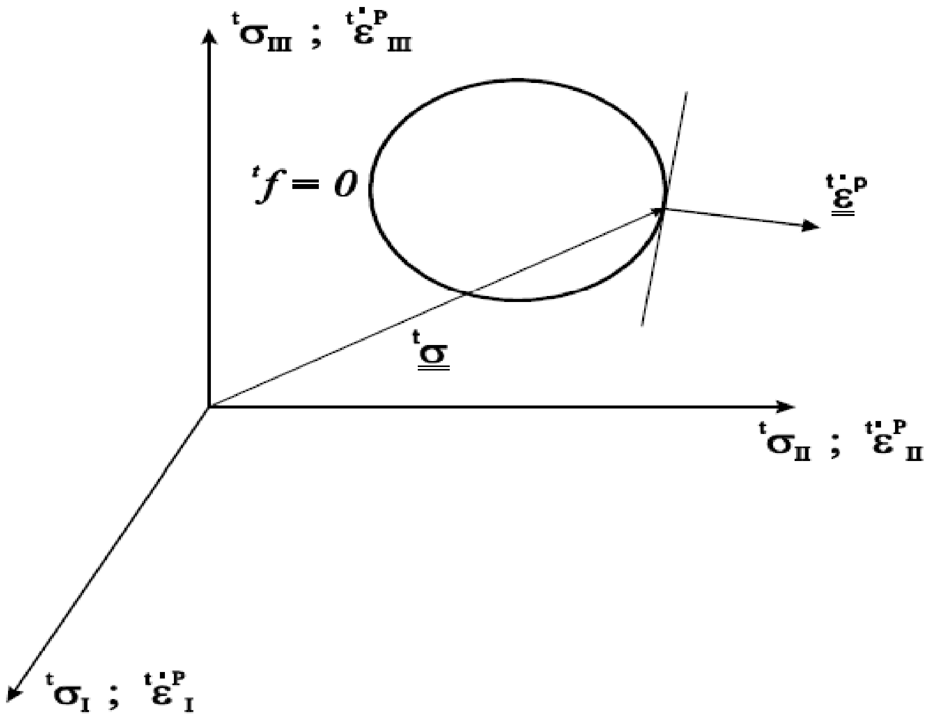


Fig. 5.6. Normality rule in associated plasticity (maximum plastic dissipation)

Drucker's postulate

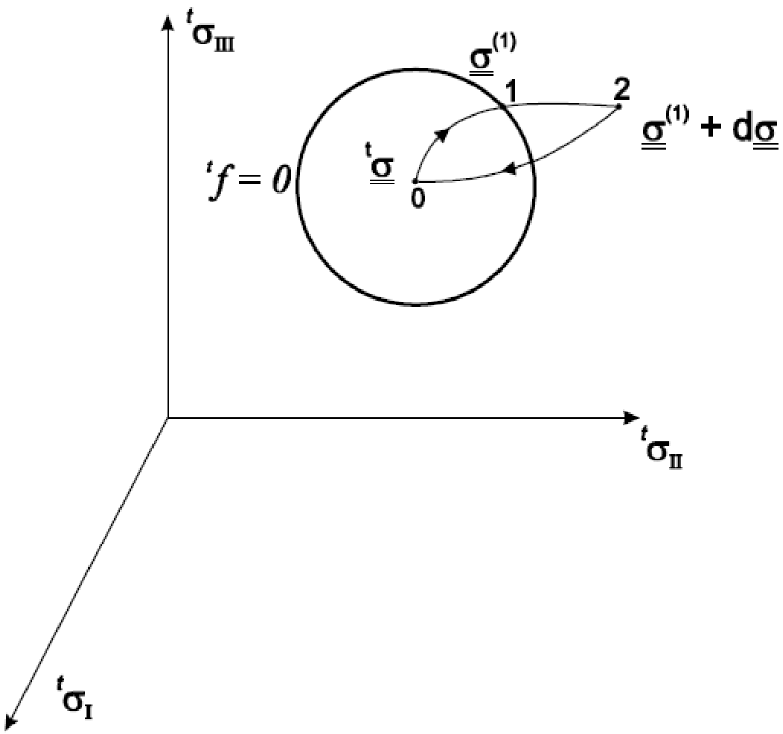


Fig. 5.7. Drucker's postulate (work-hardening material)

Trajectory	$d\varepsilon_{ij}$
0 - 1	$d\varepsilon_{ij}^{E}$
1 - 2	$d\varepsilon_{ij}^{E} + d\varepsilon_{ij}^{P}$
2 - 0	$d\varepsilon_{ij}^{E}$

Drucker's postulate

The total work per unit volume performed during the cycle 0-1-2-0 is,

$$W_{TOTAL} = \int_0^1 \sigma_{ij} \;\; \mathrm{l}\varepsilon_{ij} + \int_1^2 \sigma_{ij} \;\; \mathrm{l}\varepsilon_{ij} + \int_2^0 \sigma_{ij} \;\; \mathrm{l}\varepsilon_{ij} \qquad (5.63a)$$

hence, the work performed by the external agent is,

$$W_{e.a} = \int_0^1 (\sigma_{ij} \; - \; {}^t\sigma_{ij} \;\; d\varepsilon_{ij} + \int_1^2 (\sigma_{ij} \; - \; {}^t\sigma_{ij}) \; d\varepsilon_{ij}$$

$$+ \int_2^0 (\sigma_{ij} \; - \; {}^t\sigma_{ij}) \; d\varepsilon_{ij} \; . \qquad (5.63b)$$

Drucker's postulate

Taking into account that

$$\oint \sigma_{ij} \; d\varepsilon_{ij}^{E} = 0$$

$$\oint {}^{t}\sigma_{ij} \; d\varepsilon_{ij}^{E} = 0$$

we get,

$$W_{e.a} = \int_{1}^{2} (\sigma_{ij} - {}^{t}\sigma_{ij}) \; d\varepsilon_{ij}^{P} \; .$$

For a stable material

$$W_{e.a} \geq 0 \; .$$

$$(\sigma_{ij} - {}^{t}\sigma_{ij}) \; d\varepsilon_{ij}^{P} \geq 0 \; .$$

Drucker's postulate

$$d\sigma_{ij} \; d\varepsilon_{ij}^{P} \;\; \geq \; 0 \; . \tag{5.63h}$$

The above constrain has already been derived for the 1D case, see Eq. (5.47).

Equations (5.63g) and (5.63h) are two mathematical constraints that a stable material has to fulfill.

We can rewrite Eq. (5.63h) as,

$$^{t}\dot{\sigma}_{ij} \;\; ^{t}d_{ij}^{P} \; \geq 0 \; . \tag{5.64}$$

The above equation indicates that the plastic strain rate cannot oppose the stress rate (Lubliner 1990).

Drucker's postulate

No nonconvex yield surfaces for stable materials

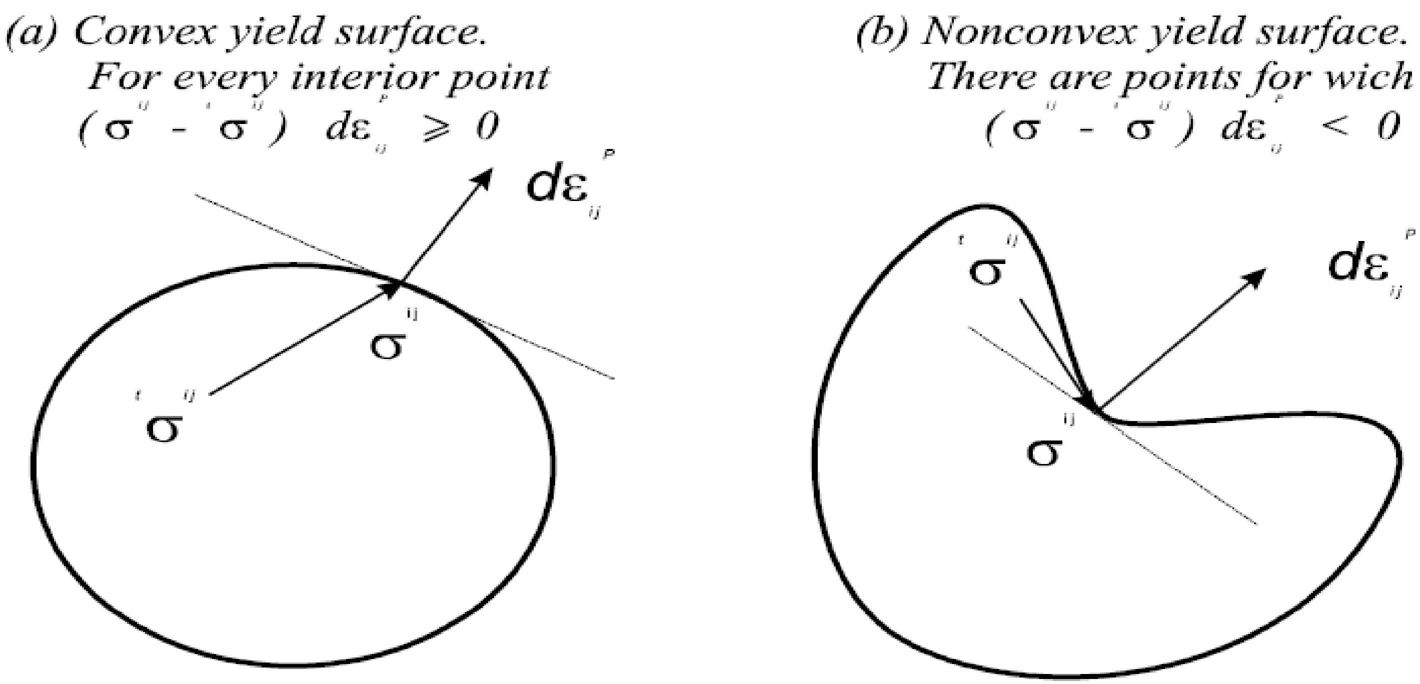


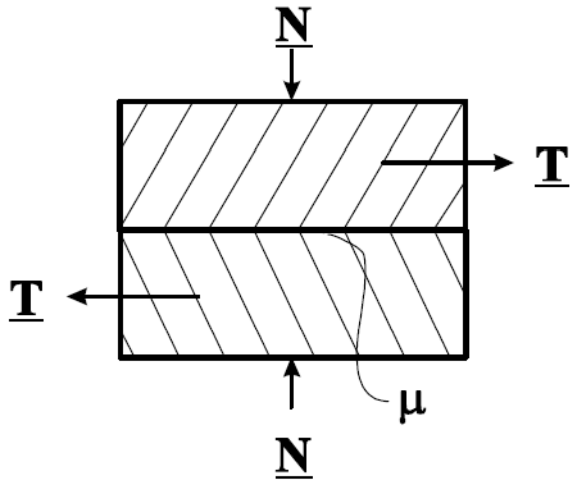
Fig. 5.8. Convexity of the yield surface as a consequence of Drucker's postulate

Limitations of associated plasticity: frictional materials (Bazant)

We formulate a yield function in the stress space using T and N (the modulus of $\underline{\mathbf{T}}$ and $\underline{\mathbf{N}}$) as independent variables,

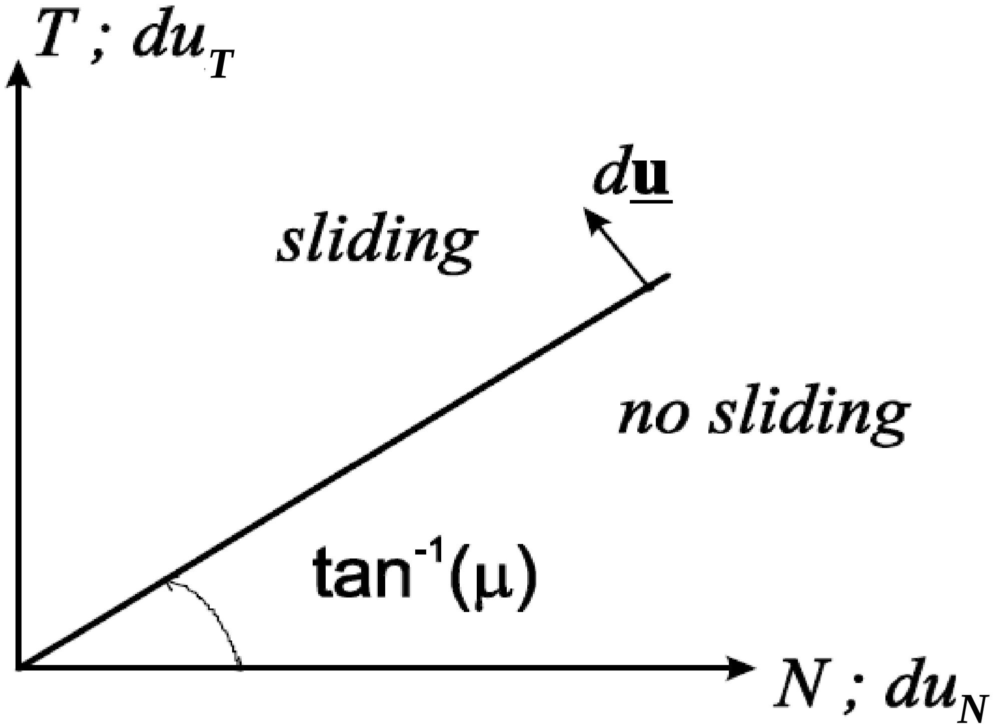
$$ {}^{t}f = T - \mu N \ , $$

when ${}^{t}f < 0 \quad \longrightarrow \quad$ *no sliding* and when ${}^{t}f = 0 \quad \longrightarrow \quad$ *sliding.*



The simplest frictional material

Limitations of associated plasticity: frictional materials (Bazant)



Yield function and plastic deformation predicted using an associated plasticity formulation (simplest frictional material)

Limitations of associated plasticity: frictional materials (Bazant)

The hypothesis of associated plasticity produces a nonphysical plastic displacement component in the $\underline{\mathbf{N}}$-direction (remember that the plates were assumed to be rigid).
Therefore, to model frictional materials, it is necessary to use nonassociated plasticity formulations (Bažant 1979, Vermeer & de Borst 1984, Dvorkin, Cuitiño & Gioia 1989). ◀◀◀◀◀

Stress-strain relations

$$\mathrm{d}\varepsilon_{\alpha\beta} = \mathrm{d}\varepsilon^{E}_{\alpha\beta} + \mathrm{d}\varepsilon^{P}_{\alpha\beta}$$

$$\mathrm{d}\sigma_{\alpha\beta} = \; C^{E}_{\alpha\beta\gamma\delta} \; \mathrm{d}\varepsilon^{E}_{\gamma\delta}$$

$$\mathrm{d}\varepsilon^{P}_{\alpha\beta} = \mathrm{d}\lambda \frac{\partial\, {}^{t}f}{\partial\, {}^{t}\sigma_{\alpha\beta}} = \mathrm{d}\lambda \frac{\partial\, {}^{t}f}{\partial\, {}^{t}s_{\gamma\delta}} \frac{\partial\, {}^{t}s_{\gamma\delta}}{\partial\, {}^{t}\sigma_{\alpha\beta}}$$

$$\frac{\partial\, {}^{t}f}{\partial\, {}^{t}s_{\gamma\delta}} = \left({}^{t}s_{\gamma\delta} - \; {}^{t}\alpha_{\gamma\delta} \right) \qquad \frac{\partial\, {}^{t}s_{\gamma\delta}}{\partial\, {}^{t}\sigma_{\alpha\beta}} = \delta_{\gamma\alpha}\delta_{\delta\beta} - \frac{1}{3}\delta_{\alpha\beta}\delta_{\gamma\delta}$$

Stress-strain relations

$$\mathrm{d}\varepsilon^P_{\alpha\beta} = \mathrm{d}\lambda \left({}^t s_{\alpha\beta} - {}^t \alpha_{\alpha\beta} \right)$$

$$\mathrm{d}\sigma_{\alpha\beta} = \quad C^E_{\alpha\beta\gamma\delta} \left[\mathrm{d}\varepsilon_{\gamma\delta} - \mathrm{d}\lambda \left({}^t s_{\gamma\delta} - {}^t \alpha_{\gamma\delta} \right) \right]$$

During plastic loading: $\mathrm{d}f = 0$

$$\frac{\partial^t f}{\partial^t \sigma_{\alpha\beta}} \mathrm{d}\sigma_{\alpha\beta} + \frac{\partial^t f}{\partial^t \varepsilon^P_{\alpha\beta}} \mathrm{d}\varepsilon^P_{\alpha\beta} = 0$$

$$\left({}^t s_{\alpha\beta} - {}^t \alpha_{\alpha\beta} \right) \; C^E_{\alpha\beta\gamma\delta} \; \left[\mathrm{d}\varepsilon_{\gamma\delta} - \; \mathrm{d}\lambda \left({}^t s_{\gamma\delta} - {}^t \alpha_{\gamma\delta} \right) \right] + \frac{\partial^t f}{\partial^t \varepsilon^P_{\alpha\beta}} \mathrm{d}\lambda \left({}^t s_{\alpha\beta} - {}^t \alpha_{\alpha\beta} \right) = 0 \; .$$

Stress-strain relations

Hence,

$$\mathrm{d}\lambda = \frac{\left({}^{t}s_{\alpha\beta} - {}^{t}\alpha_{\alpha\beta} \right) C^{E}_{\alpha\beta\gamma\delta}\, \mathrm{d}\varepsilon_{\gamma\delta}}{\left({}^{t}s_{\varepsilon\zeta} - {}^{t}\alpha_{\varepsilon\zeta} \right) C^{E}_{\varepsilon\zeta\eta\vartheta} \left({}^{t}s_{\eta\vartheta} - {}^{t}\alpha_{\eta\vartheta} \right) - \frac{\partial^{t} f}{\partial^{t}\varepsilon^{P}_{\varepsilon\zeta}} \left({}^{t}s_{\varepsilon\zeta} - {}^{t}\alpha_{\varepsilon\zeta} \right)} \,.$$

$$\mathrm{d}\sigma_{\alpha\beta} = \Bigg[C^{E}_{\alpha\beta\gamma\delta} - \tag{5.69}$$

$$\frac{\left({}^{t}s_{\nu\mu} - {}^{t}\alpha_{\nu\mu} \right) C^{E}_{\alpha\beta\nu\mu}\, C^{E}_{\varphi\xi\gamma\delta} \left({}^{t}s_{\varphi\xi} - {}^{t}\alpha_{\varphi\xi} \right)}{\left({}^{t}s_{\rho\pi} - {}^{t}\alpha_{\rho\pi} \right) C^{E}_{\rho\pi\eta\tau} \left({}^{t}s_{\eta\tau} - {}^{t}\alpha_{\eta\tau} \right) - \frac{\partial^{t} f}{\partial^{t}\varepsilon^{P}_{\rho\pi}} \left({}^{t}s_{\rho\pi} - {}^{t}\alpha_{\rho\pi} \right)} \Bigg] \mathrm{d}\varepsilon_{\gamma\delta} \,.$$

Stress-strain relations

$$d\underline{\underline{\sigma}} = {}^{t}\underline{\underline{\underline{\underline{C}}}}^{EP} : d\underline{\underline{\varepsilon}}$$

- ${}^{t}C^{EP}_{\alpha\beta\gamma\delta} = {}^{t}C^{EP}_{\beta\alpha\gamma\delta}$
- ${}^{t}C^{EP}_{\alpha\beta\gamma\delta} = {}^{t}C^{EP}_{\alpha\beta\delta\gamma}$
- ${}^{t}C^{EP}_{\alpha\beta\gamma\delta} = {}^{t}C^{EP}_{\gamma\delta\alpha\beta}$

It is important to realize that the last symmetry is lost in the case of non-associated plastic models (see Eq. (5.62b)) (Bažant 1979, Vermeer & de Borst 1984).

Hardening laws: Isotropic hardening

$$ {}^{t}\underline{\underline{\alpha}} = \underline{\underline{0}} $$

$$ {}^{t}\sigma_y = {}^{t}\sigma_y(\,{}^{t}W^P)\;. $$

$$ {}^{t}W = \int_0^{{}^{t}\varepsilon_{\gamma\delta}} \sigma_{\alpha\beta}\; \mathrm{d}\varepsilon_{\alpha\beta} $$

$$ {}^{t}W = \int_0^{{}^{t}\varepsilon^E_{\gamma\delta}} \sigma_{\alpha\beta}\; \mathrm{d}\varepsilon^E_{\alpha\beta} + \int_0^{{}^{t}\varepsilon^P_{\gamma\delta}} \sigma_{\alpha\beta}\; \mathrm{d}\varepsilon^P_{\alpha\beta} $$

Hardening laws: Isotropic hardening

$$ {}^{t}\overline{\sigma} = {}^{t}\sigma_{y}({}^{t}W^{P}) $$

$$ {}^{t}\overline{\sigma} : \text{equivalent stress} = \sqrt{\frac{3}{2}\, {}^{t}\underline{\underline{\mathbf{s}}} : {}^{t}\underline{\underline{\mathbf{s}}}} $$

$$ {}^{t}\dot{W}^{P} = {}^{t}s_{\alpha\beta}\, {}^{t}d^{P}_{\alpha\beta} $$

$$ {}^{t}\overline{d}^{P} = \sqrt{\frac{2}{3}\, {}^{t}\underline{\underline{\mathbf{d}}}^{P} : {}^{t}\underline{\underline{\mathbf{d}}}^{P}} $$

Hardening laws: Isotropic hardening

Hence,

$$ {}^{t}\overline{\sigma}\ {}^{t}\overline{d}^{P} = \sqrt{\left({}^{t}\underline{\underline{\mathrm{s}}} : {}^{t}\underline{\underline{\mathrm{s}}} \right) \left({}^{t}\underline{\underline{\mathbf{d}}}^{P} : {}^{t}\underline{\underline{\mathbf{d}}}^{P} \right)}\ . \tag{5.73b} $$

Using Eq. (5.60) we can see that ${}^{t}\underline{\underline{\mathbf{d}}}^{P}$ and ${}^{t}\underline{\underline{\mathrm{s}}}$ are collinear tensors when isotropic von Mises plasticity is considered; hence, we can write Eq. (5.73b) as,

$$ {}^{t}\overline{\sigma}\ {}^{t}\overline{d}^{P} = {}^{t}\underline{\underline{\mathrm{s}}} : {}^{t}\underline{\underline{\mathbf{d}}}^{P} = {}^{t}\dot{W}^{P} \tag{5.73c} $$

$$ {}^{t}\overline{\varepsilon}^{P} = \int_{\underline{\underline{\mathbf{0}}}}^{{}^{t}\underline{\underline{\varepsilon}}^{P}} {}^{t}\overline{d}^{P}\ \mathrm{d}t $$

$$ \mathrm{d}W^{P} = {}^{t}\sigma_{y}\left({}^{t}W^{P} \right) \mathrm{d}\overline{\varepsilon}^{P} $$

Hardening laws: Isotropic hardening

$$\mathrm{d}\sigma_y = \frac{\partial^t\sigma_y}{\partial^t W^P}\,{}^t\sigma_y\,\mathrm{d}\overline{\varepsilon}^P = \frac{\partial^t\sigma_y}{\partial^t W^P}\frac{\partial^t W^P}{\partial^t\overline{\varepsilon}^P}\,\mathrm{d}\overline{\varepsilon}^P = \frac{\partial^t\sigma_y}{\partial^t\overline{\varepsilon}^P}\,\mathrm{d}\overline{\varepsilon}^P$$

Therefore, we can construct a curve,

$$ {}^t\sigma_y = {}^t\sigma_y\left({}^t\overline{\varepsilon}^P\right)\ . \tag{5.75}$$

"*The assumption that one universal stress - strain curve of the form of Eq. (5.75) governs all possible combined - stress loadings of a given material is obviously a very strong one*" (Malvern 1969).

Isotropic hardening: the tensile test

Example 5.12. ◀◀◀◀◀

In a uniaxial tensile test, before the necking is localized, with the I-axis the loading direction and the II and III - axes orthogonal ones, we can write

$$
\begin{aligned}
{}^{t}\sigma_{I} &= \sigma^{*} \quad , \quad & {}^{t}s_{I} &= \frac{2}{3}\sigma^{*} \; , \\
{}^{t}\sigma_{II} &= 0 \quad , \quad & {}^{t}s_{II} &= -\frac{1}{3}\sigma^{*} \; , \\
{}^{t}\sigma_{III} &= 0 \quad , \quad & {}^{t}s_{III} &= -\frac{1}{3}\sigma^{*} \; .
\end{aligned}
$$

Hence,

$$\overline{\sigma} = \sigma^{*} \; .$$

Also, due to incompressibility

Isotropic hardening: the tensile test

$$
\begin{aligned}
{}^t\varepsilon_I^P &= \varepsilon_P^* \\
{}^t\varepsilon_{II}^P &= -\frac{1}{2}\varepsilon_P^* \\
{}^t\varepsilon_{III}^P &= -\frac{1}{2}\varepsilon_P^* \ .
\end{aligned}
$$

Hence,

$$
{}^t\overline{\varepsilon}^P = \varepsilon_P^* \ .
$$

Therefore, for an isotropic hardening von Mises material, the complete universal stress - strain curve is determined with only an uniaxial test. ◀◀◀◀◀

Hardening laws: kinematic hardening

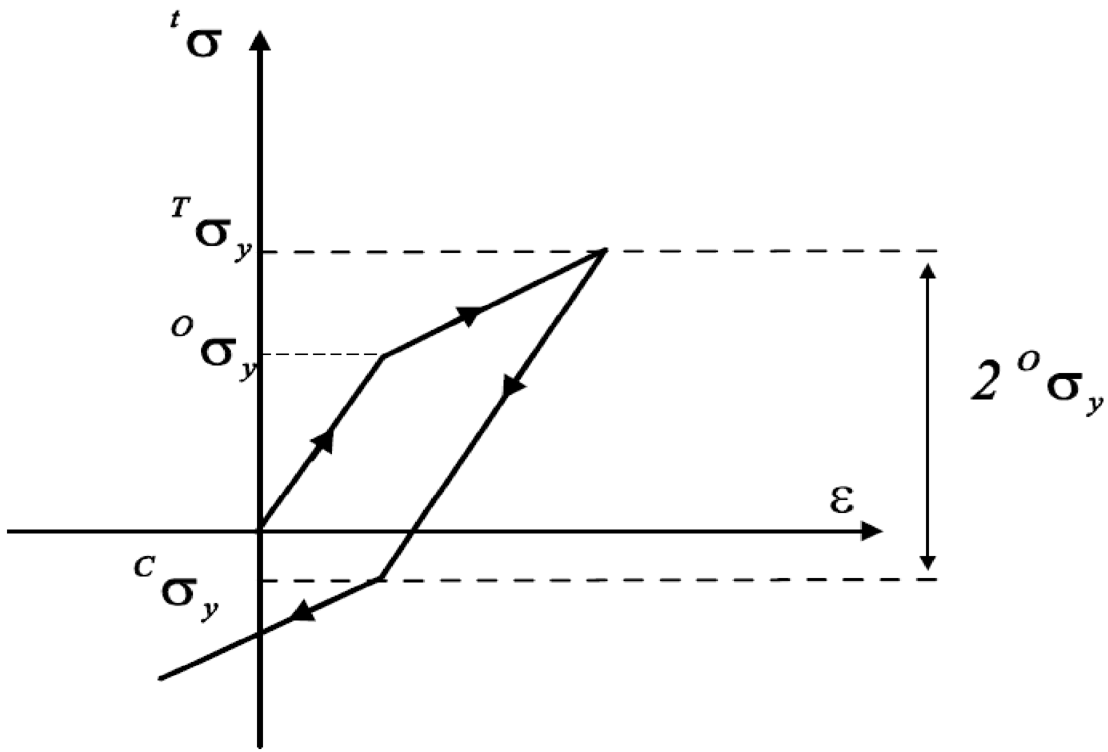


Fig. 5.9. Bauschinger effect

Hardening laws: kinematic hardening

$$^{t}\sigma_y = \,^{\circ}\sigma_y = const.$$

Prager, in his kinematic hardening model, assumes a linear hardening (Malvern 1969):

$$^{t}\dot{\alpha}_{ij} = c \;\; ^{t}d^{P}_{ij} \tag{5.77}$$

$$^{t}d^{P}_{kk} = 0 \;\Rightarrow\; ^{t}\dot{\alpha}_{kk} = 0$$

Finite elastoplastic strains
Lee's multiplicative decomposition

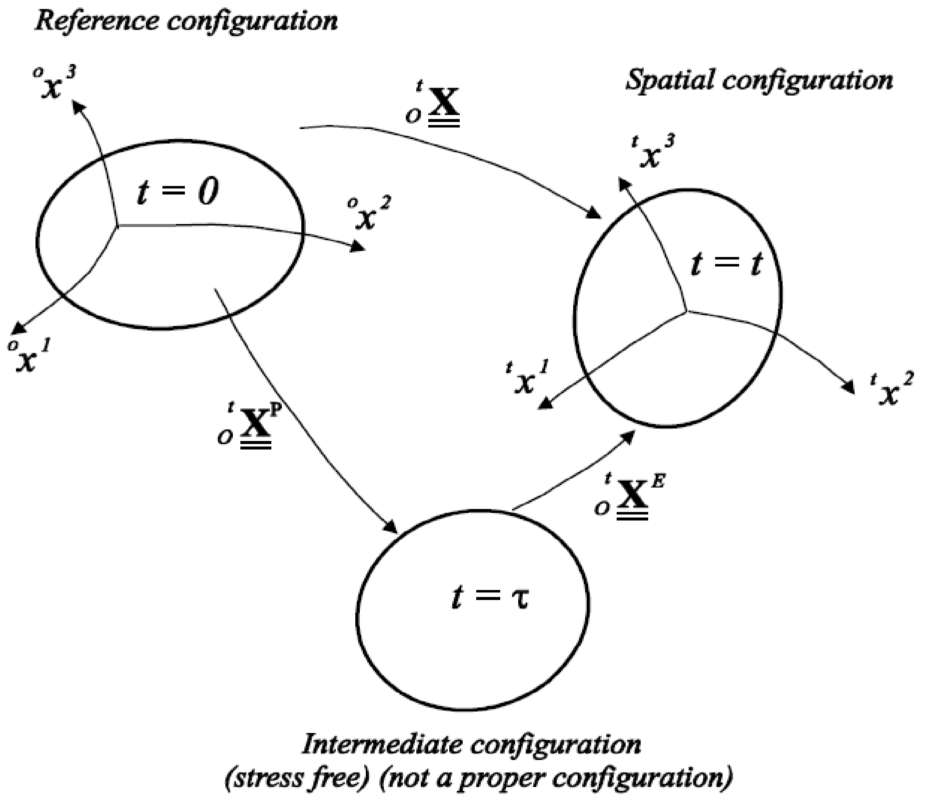


Fig. 5.10. Lee's multiplicative decomposition of the deformation gradient

Finite elastoplastic strains
Lee's multiplicative decomposition

$$ {}^{t}_{\circ}\underline{\underline{\mathbf{X}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{P} $$

$$ {}^{t}\underline{\underline{\mathbf{l}}} = {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{X}}}} \cdot {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{-1} $$

$$ {}^{t}\underline{\underline{\mathbf{l}}} = {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{X}}}}^{E} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E}\right)^{-1} + {}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E} \cdot {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{X}}}}^{P} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{P}\right)^{-1} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{E}\right)^{-1} $$

$$ {}^{t}\bar{\underline{\underline{\mathbf{l}}}}^{P} = {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{X}}}}^{P} \cdot \left({}^{t}_{\circ}\underline{\underline{\mathbf{X}}}^{P}\right)^{-1} $$

Finite elastoplastic strains
Lee's multiplicative decomposition

It is not as simple as:

$$ {}^{t}\underline{\underline{\mathbf{d}}} = {}^{t}\underline{\underline{\mathbf{d}}}^{E} + {}^{t}\underline{\underline{\mathbf{d}}}^{P} $$

Viscoplasticity

Plasticity and Viscoplasticity

Instantaneous Plasticity

$$\sigma_y = \sigma_y(\bar{\varepsilon}, T)$$

Viscoplasticity

$$\sigma_y = \sigma_y(\bar{\varepsilon}, \dot{\bar{\varepsilon}}, T)$$

$$^{t}\underline{\underline{\mathbf{d}}} = {}^{t}\underline{\underline{\mathbf{d}}}^{E} + {}^{t}\underline{\underline{\mathbf{d}}}^{VP}$$

Rigid – viscoplastic models

In some cases, for example when modeling bulk metal-forming processes (Zienkiewicz, Jain & Oñate 1977), ${}^{t}\underline{\underline{\mathbf{d}}}^{E} << {}^{t}\underline{\underline{\mathbf{d}}}^{VP}$. Therefore, we can set ${}^{t}\underline{\underline{\mathbf{d}}}^{E} = \underline{\underline{\mathbf{0}}}$, introducing a very important simplification in the model without any significant loss in accuracy; these are the *rigid-viscoplastic* material models.

SIM&TEC
Simulación y Tecnología
Simulation and Technology

The yield surface

$$ {}^{t}f\left({}^{t}\underline{\underline{\sigma}},\ {}^{t}q_i \quad i=1,n\right)=0 $$

${}^{t}f < 0$	${}^{t}\underline{\underline{d}}^{VP} = 0$
${}^{t}f \geq 0$	${}^{t}\underline{\underline{d}}^{VP} \neq 0$

Perzyna's flow rule

$$ {}^{t}d^{VP}_{\alpha\beta} = \gamma \frac{\partial\, {}^{t}f}{\partial\, {}^{t}\sigma_{\alpha\beta}} \left\langle \phi \left({}^{t}f \right) \right\rangle $$

In the above equation, we use the Macauley brackets defined by:

$$ \langle a \rangle = a \quad if\ a > 0 \qquad (5.161a) $$

$$ \langle a \rangle = 0 \quad if\ a \leq 0\ . \qquad (5.161b) $$

${}^{t}f$: von Mises function

Perzyna's flow rule

An important difference between the flow rate for the viscoplastic constitutive model (Eq. (5.160)) and the flow rate for the plastic constitutive model (Eq. (5.60)) is that in the present case, γ the *fluidity parameter* is a material constant, while in the plasticity theory ${}^{t}\dot{\lambda}$ is a flow constant, derived by imposing the consistency condition during the plastic loading.

Obviously, the correct value of γ and the correct expression for $\phi\ ({}^{t}f)$ are derived from experimental observations.

In what follows we will concentrate on the details of a rigid-viscoplastic relation suited for describing the behavior of metals with isotropic hardening,

$$\phi({}^{t}f) = \left[\underbrace{\left(\frac{1}{2}\ {}^{t}s_{\alpha\beta}\ {}^{t}s_{\alpha\beta} \right)^{\frac{1}{2}} - \frac{{}^{t}\sigma_y}{\sqrt{3}}}_{{}^{t}f} \right]^{\delta} . \qquad (5.162)$$

Perzyna's flow rule

$$\left.\frac{\partial f}{\partial \sigma_{\alpha\beta}}\right|_t = \frac{1}{2\sqrt{{}^tJ_2}}\ {}^ts_{\alpha\beta}$$

$${}^td^{VP}_{\alpha\beta} = \frac{\gamma}{2\sqrt{{}^tJ_2}}\ {}^ts_{\alpha\beta}\ \langle\ {}^tf^{\delta}\ \rangle$$

$${}^t\dot{\bar{\varepsilon}}_{VP} = \frac{\gamma}{\sqrt{3}}\langle\ {}^tf^{\delta}\ \rangle \qquad \left({}^tf\right)^{\delta} = \frac{\sqrt{3}}{\gamma}\ {}^t\dot{\bar{\varepsilon}}_{VP}$$

Perzyna's flow rule
Rigid- viscoplastic model

$$ {}^t s_{\alpha\beta} = 2 {}^t\mu \; {}^t d_{\alpha\beta}^{VP} $$

$$ {}^t\mu = \frac{\dfrac{{}^t\sigma_y}{\sqrt{3}} + \left[\dfrac{\sqrt{3}}{\gamma} \; {}^t\dot{\bar{\varepsilon}}_{VP}\right]^{\frac{1}{\delta}}}{\sqrt{3} \; {}^t\dot{\bar{\varepsilon}}_{VP}} $$

In the limit, when $\gamma \to \infty$ Eq. (5.168) describes the behavior of a rigid-plastic material (inviscid), in this case,

$$ {}^t\mu = \frac{{}^t\sigma_y}{3 \; {}^t\dot{\bar{\varepsilon}}_{VP}} \; . \qquad (5.169) $$

Dynamic loading

Example 5.16. ◀◀◀◀◀

An important experimentally observed effect, that the viscoplastic material model explains, is the increase in the apparent yield stress of metals when the strain rate is increased (Malvern 1969) (strain-rate effect).

Let us assume a uniaxial test in a rigid-viscoplastic bar,

$$\sigma_{11} = \widehat{\sigma}$$
$$\sigma_{22} = \sigma_{33} = 0 \cdot$$

Therefore,

$$s_{11} = \frac{2}{3}\widehat{\sigma}$$
$$s_{22} = s_{33} = -\frac{1}{3}\widehat{\sigma} \cdot$$

Also, for the viscoplastic strain rates we can write,

$$d_{11}^{VP} = \dot{\varepsilon}$$
$$d_{22}^{VP} = d_{33}^{VP} = -\frac{1}{2}\dot{\varepsilon} \cdot$$

Hence, the equivalent viscoplastic strain rate is,

$$\dot{\bar{\varepsilon}}_{VP} = \dot{\varepsilon} \ .$$

Dynamic loading

Get,

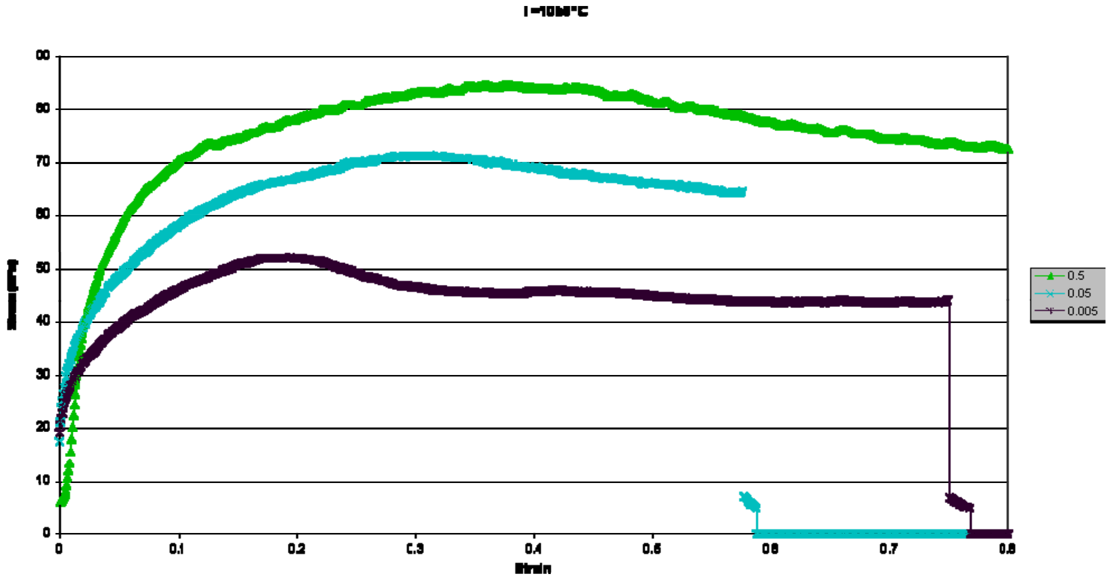
$$\widehat{\sigma} = \sigma_y + \sqrt{3}\left(\frac{\sqrt{3}}{\gamma}\dot{\varepsilon}\right)^{1/\delta}$$

When $\gamma \rightarrow \infty$ $\hat{\sigma} \rightarrow \sigma_y$

Phenomenological constitutive relations

Example: modeling hot metal forming

Material properties



Compression tests at elevated temperature showing recristallization

Phenomenological constitutive relations

Example: modeling hot metal forming

1. The Fields – Backofen law

$$\sigma_y = A(T)\,\bar{\varepsilon}^{n(T)}\,\dot{\bar{\varepsilon}}^{m(T)}$$

This model cannot represent recristalization phenomena

2. Exponential – power law 1

$$\sigma_y = [A(T)\,e^{-B(T)\bar{\varepsilon}}\,(\bar{\varepsilon} + \bar{\varepsilon}_o)^{n(T)} + C(T)\,(1 - e^{-B(T)\bar{\varepsilon}})]\,\dot{\bar{\varepsilon}}^{m(T)}$$

3. Exponential – power law 2

$$\sigma_y = [A(T)\,e^{-B(T)\bar{\varepsilon}}\,\sqrt{(1 - e^{-n(T)(\bar{\varepsilon}+\bar{\varepsilon}_o)})} + C(T)\,(1 - e^{-B(T)\bar{\varepsilon}})]\dot{\bar{\varepsilon}}^{m(T)}$$

Phenomenological constitutive relations

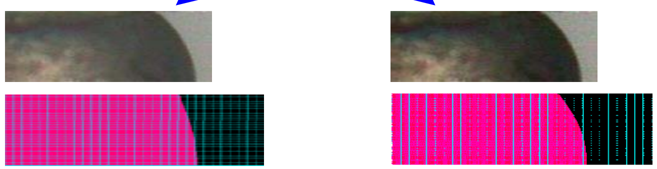
Example: modeling hot metal forming

Mechanical tests to determine the material constants

- ✓ Tension tests. Strain limitations
- ✓ Compression tests logarithmic strain < 0.8

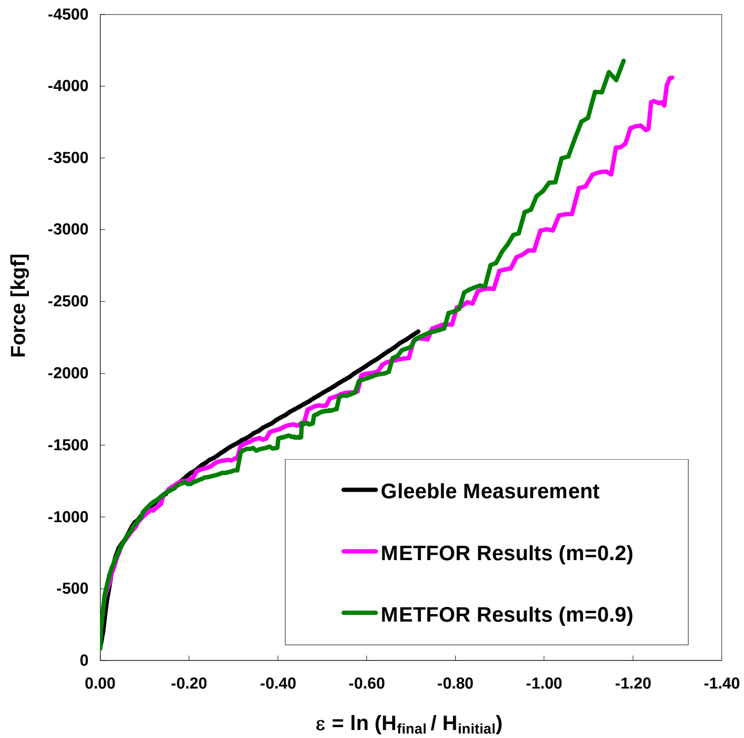
Samples barreling

Samples final shape



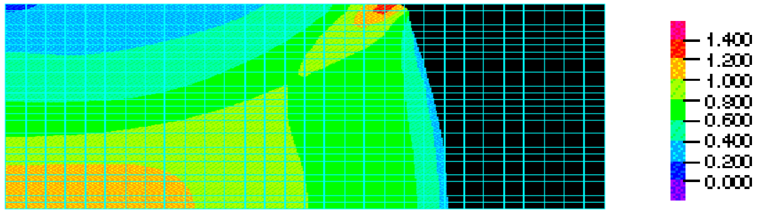
FEM
Light friction; m=0.2

FEM
Hight friction; m=0.9



Phenomenological constitutive relations

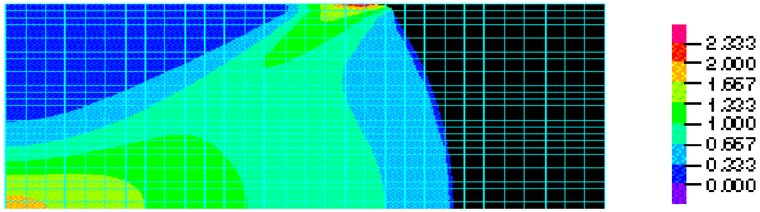
Example: modeling hot metal forming



m=0.2

Coeficiente de fricción m= 0.2

Compression tests



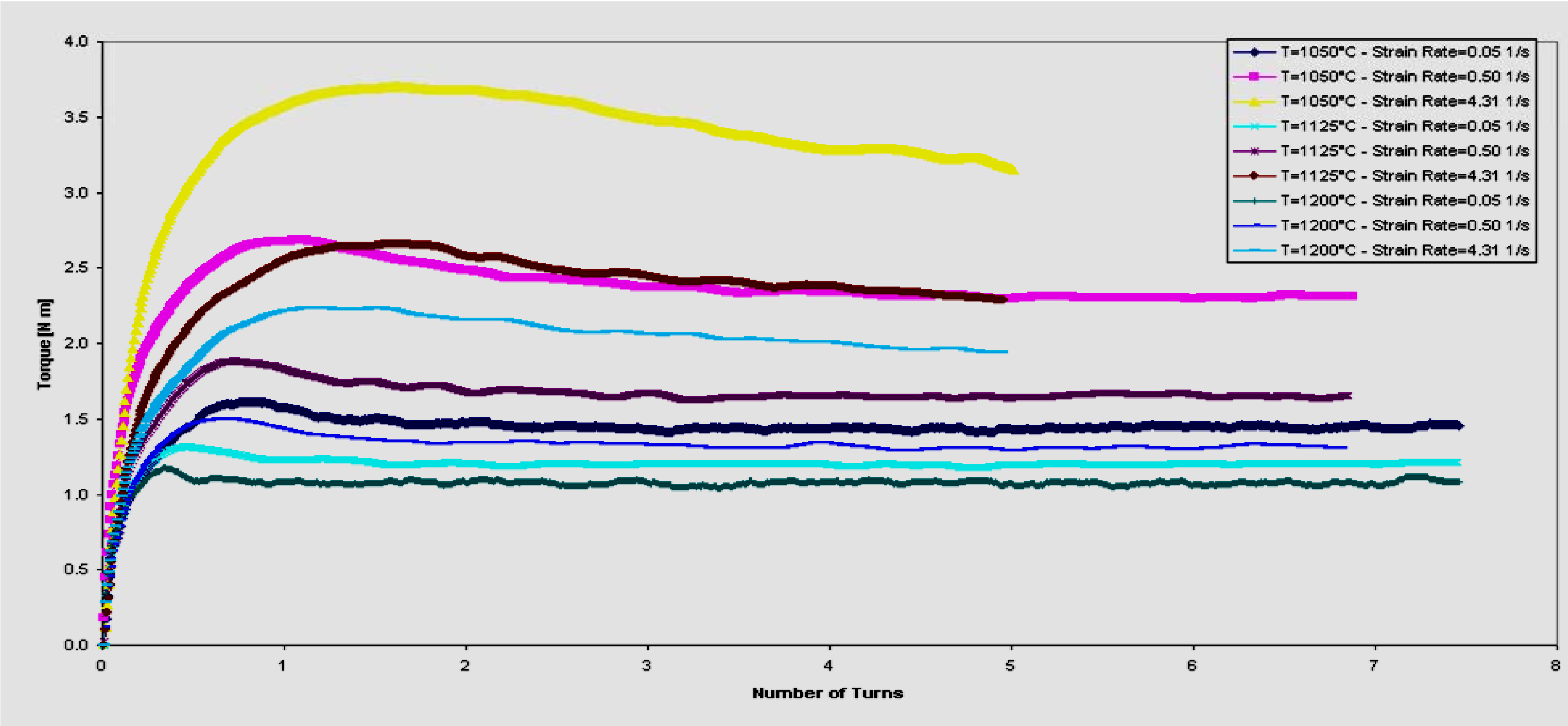
m=0.9

Coeficiente de fricción m= 0.9

When there is friction, it fails to represent a uniform strain test

Phenomenological constitutive relations

Example: modeling hot metal forming



The torsion test

Phenomenological constitutive relations

Example: modeling hot metal forming

Results obtained using TESTPOST