



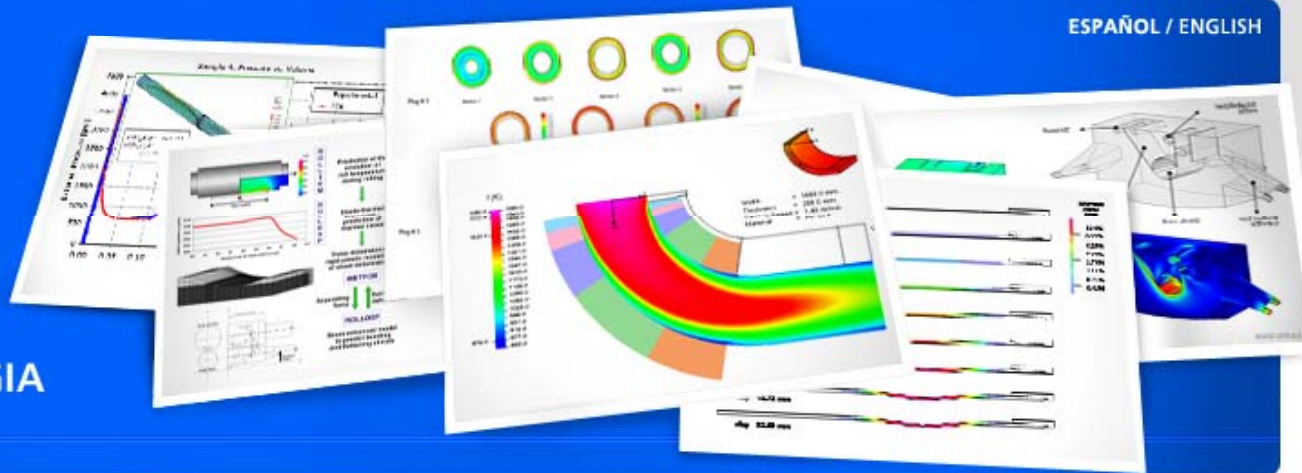
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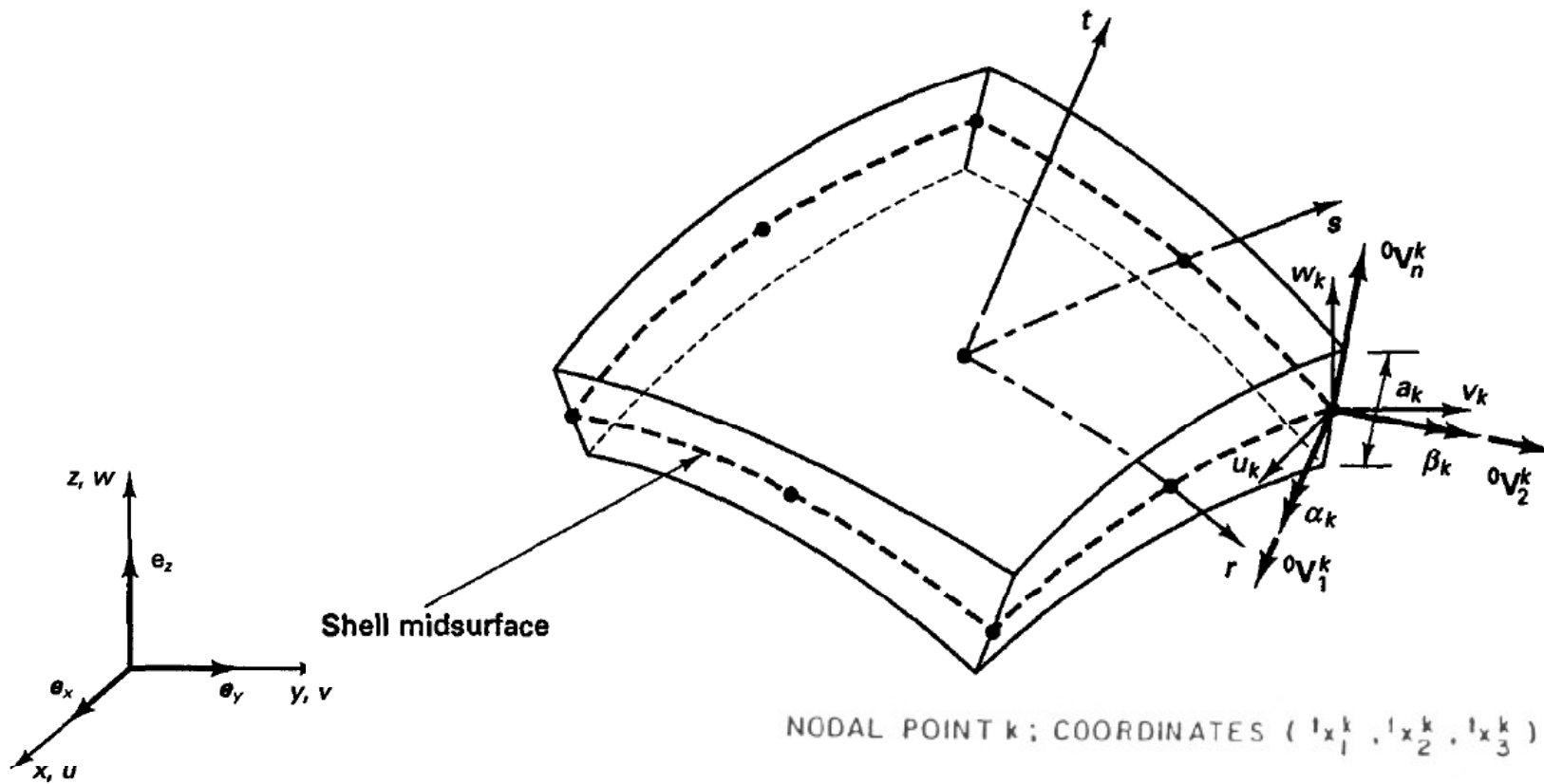
Advanced Topics in Computational Solid Mechanics. Industrial Applications

Section 8: General Nonlinear Shell Elements

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The Ahmad-Irons- Zienkiewicz Element C^0 Continuity



The Ahmad-Irons- Zienkiewicz Element C^0 Continuity

$${}^t \underline{x} = h_k {}^t x_k + \frac{r_3}{2} h_k {}^t a_k {}^t \underline{V}_n^k$$

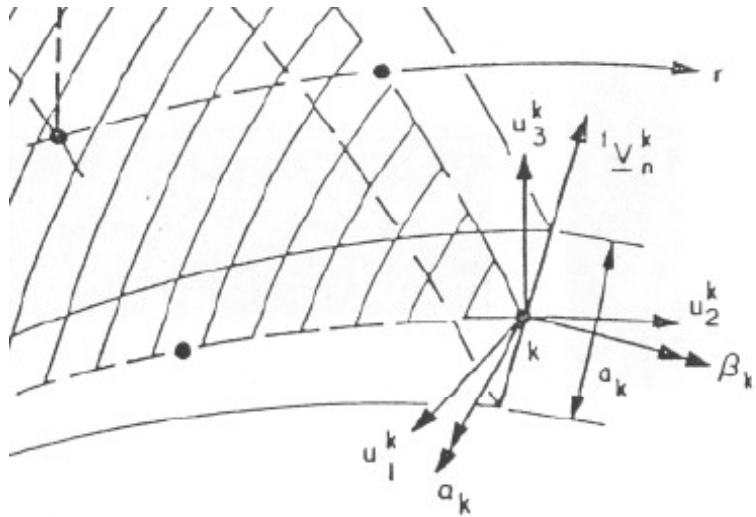
$$\left| {}^t \underline{V}_n^k \right| = 1$$

($t = 0$ represents the reference undeformed configuration)

- The director vectors remain straight during the deformation process.
- The thickness remains constant during the deformation process (${}^{t+\Delta t} a_k = {}^t a_k = \dots = {}^{\circ} a_k$).

Incremental Displacements

$$\underline{\mathbf{u}} = {}^{t+\Delta t}\underline{\mathbf{x}} - {}^t\underline{\mathbf{x}} = h_k \underline{\mathbf{u}}_k + \frac{r_3}{2} h_k {}^t a_k (-\alpha_k {}^t \underline{\mathbf{V}}_2^k + \beta_k {}^t \underline{\mathbf{V}}_1^k)$$



- If ${}^t \underline{\mathbf{e}}_2 \times {}^t \underline{\mathbf{V}}_n^k \neq \underline{\mathbf{0}}$

$${}^t \underline{\mathbf{V}}_1^k = \frac{{}^t \underline{\mathbf{e}}_2 \times {}^t \underline{\mathbf{V}}_n^k}{|{}^t \underline{\mathbf{e}}_2 \times {}^t \underline{\mathbf{V}}_n^k|}$$

$${}^t \underline{\mathbf{V}}_2^k = {}^t \underline{\mathbf{V}}_n^k \times {}^t \underline{\mathbf{V}}_1^k$$

- If ${}^t \underline{\mathbf{e}}_2 \times {}^t \underline{\mathbf{V}}_n^k = \underline{\mathbf{0}}$

$${}^t \underline{\mathbf{V}}_1^k = {}^t \underline{\mathbf{e}}_3$$

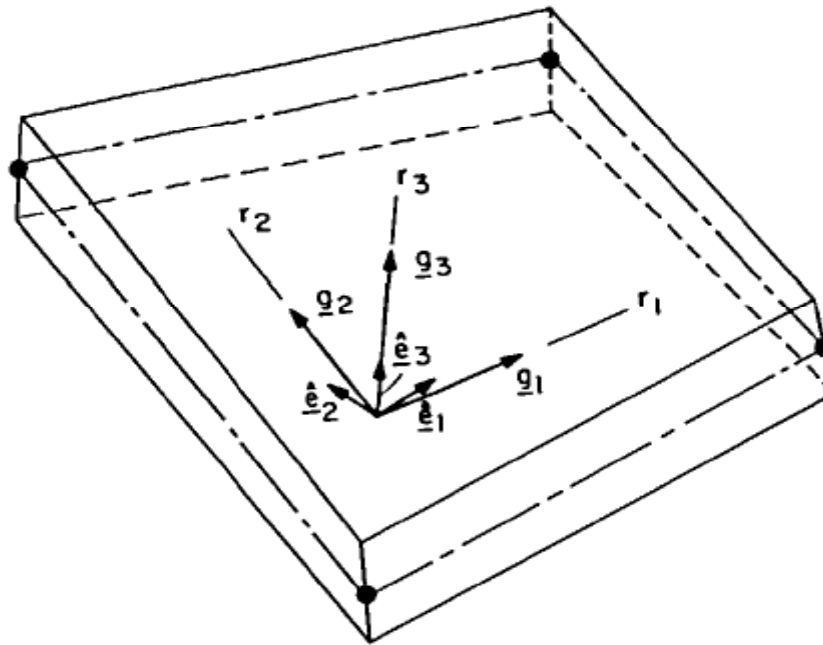
$${}^t \underline{\mathbf{V}}_2^k = {}^t \underline{\mathbf{e}}_1$$

Incremental Displacements

A special definition of $\underline{\mathbf{V}}_2^k$ and $\underline{\mathbf{V}}_1^k$ is used for the case when $\underline{\mathbf{V}}_n^k$ is parallel to the y-axis.

5 d.o.f. / node

Constitutive Relations



$$\hat{\mathbf{e}}_3 = \frac{\mathbf{g}_3}{|\mathbf{g}_3|}; \hat{\mathbf{e}}_1 = \frac{\mathbf{g}_2 \times \hat{\mathbf{e}}_3}{|\mathbf{g}_2 \times \hat{\mathbf{e}}_3|}; \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1$$

Figure 3 Local Cartesian coordinate system used

$$\hat{\sigma}_{33} = 0$$

$$\hat{\epsilon}_{33} = 0$$

$$\tilde{C}_{ijkl} = (\mathbf{g}^i \cdot \hat{\mathbf{e}}_m)(\mathbf{g}^j \cdot \hat{\mathbf{e}}_n)(\mathbf{g}^k \cdot \hat{\mathbf{e}}_o)(\mathbf{g}^l \cdot \hat{\mathbf{e}}_p) \hat{C}_{mnop}$$

$$\tilde{\mathbf{t}}_{ij} = \tilde{C}_{ijkl} \tilde{\epsilon}_{kl}$$

The Locking Problem



$$w = h_1 w_1 + h_2 w_2 \quad (\text{linear})$$

$$\theta = h_1 \theta_1 + h_2 \theta_2 \quad (\text{linear})$$

$$\underbrace{\frac{dw}{dx}}_{\text{const}} - \theta = 0$$



$$\theta = \text{const} = 0 \quad (\text{B.C.})$$

LOCKING

The Locking Problem

$$\pi = \frac{h^3}{2} \int_A {}^t \underline{x}^T \underline{C}_b {}^t \underline{x} dA + \alpha \int_A {}^t \underline{\chi}^T \underline{C}_s {}^t \underline{\chi} dA$$

$${}^t \underline{x} = \begin{bmatrix} \frac{\partial \alpha}{\partial x} \\ -\frac{\partial \beta}{\partial y} \\ \frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial x} \end{bmatrix}$$

$${}^t \underline{\chi} = \frac{1}{L} \begin{bmatrix} \frac{\partial w}{\partial y} - \beta \\ \frac{\partial w}{\partial x} + \alpha \end{bmatrix}$$

$$\underline{C}_b = \frac{E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

$$\underline{C}_s = \frac{k E}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

E : Young's modulus

ν : Poisson ratio

k : shear correction factor

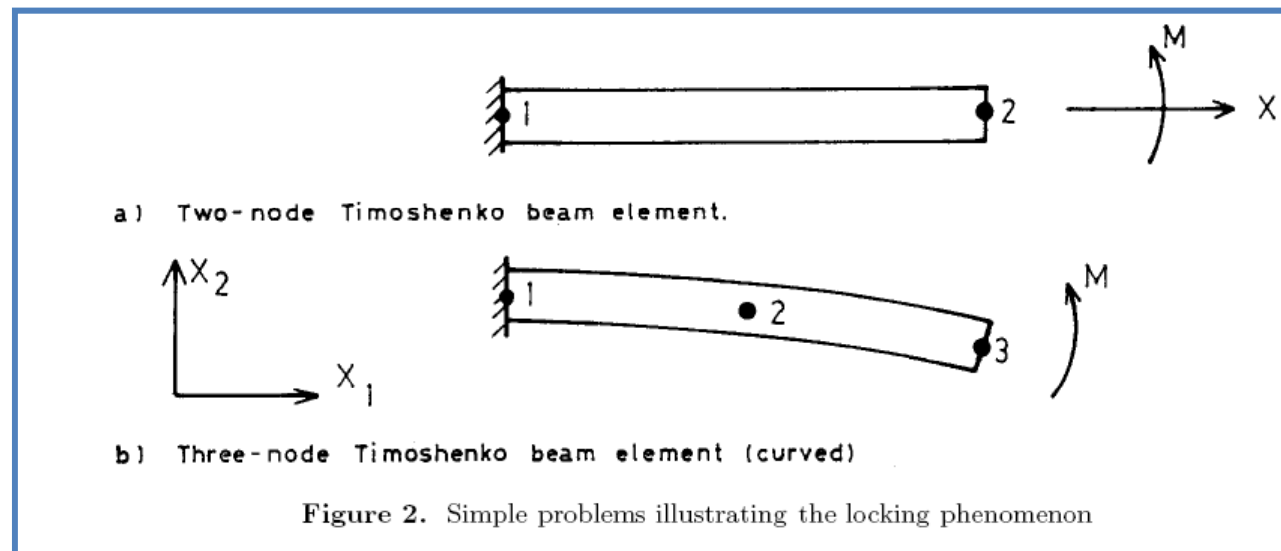
$$\alpha = \frac{L}{h} \xrightarrow{h \rightarrow 0} \omega$$

where w = transverse displacement of the plate

h = plate thickness

The Locking Problem

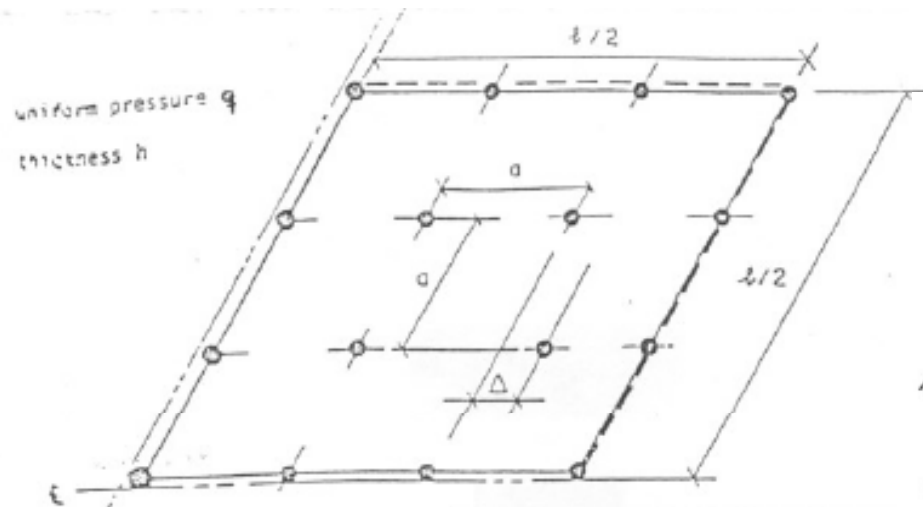
In curved elements a similar problem appears if the interpolation functions cannot represent states of zero membrane deformation (membrane locking)



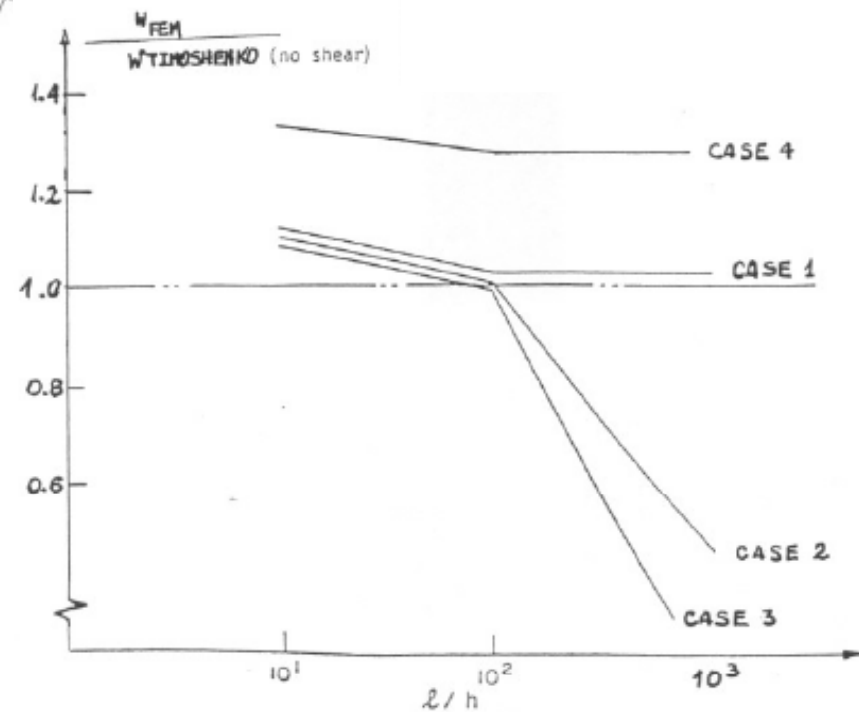
$$\pi = \frac{EI}{2} \left[\int_0^L \theta_{,s}^2 ds + \xi_M \int_0^L u_{s,s}^2 ds + \xi_S \int_0^L (u_{n,s} - \theta)^2 ds \right] - V$$

The Locking Problem

Simply-supported plate

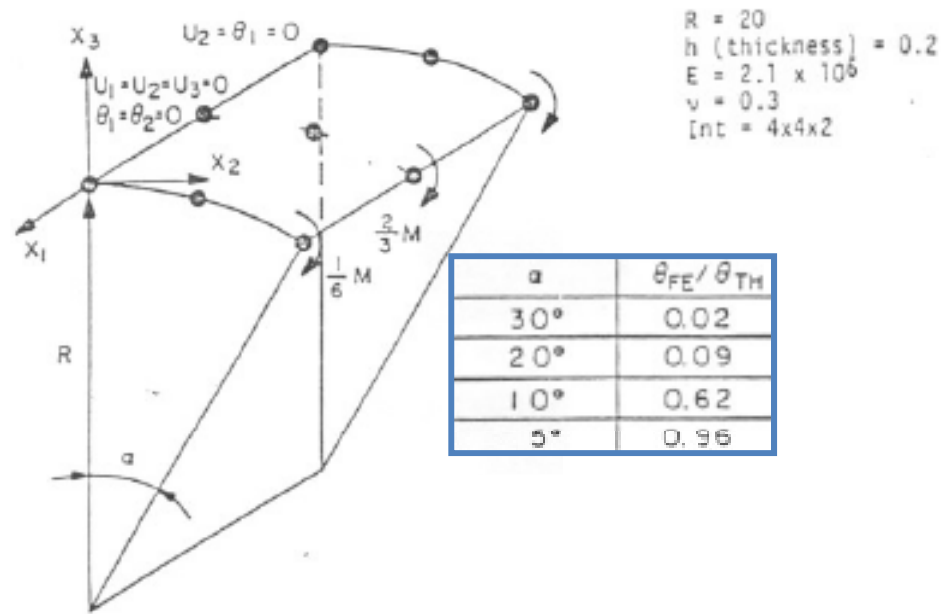


Case	Δ/a	Integration
1	0	$4 \times 4 \times 2$
2	1/50	$4 \times 4 \times 2$
3	1/20	$4 \times 4 \times 2$
4	1/20	$3 \times 3 \times 2$

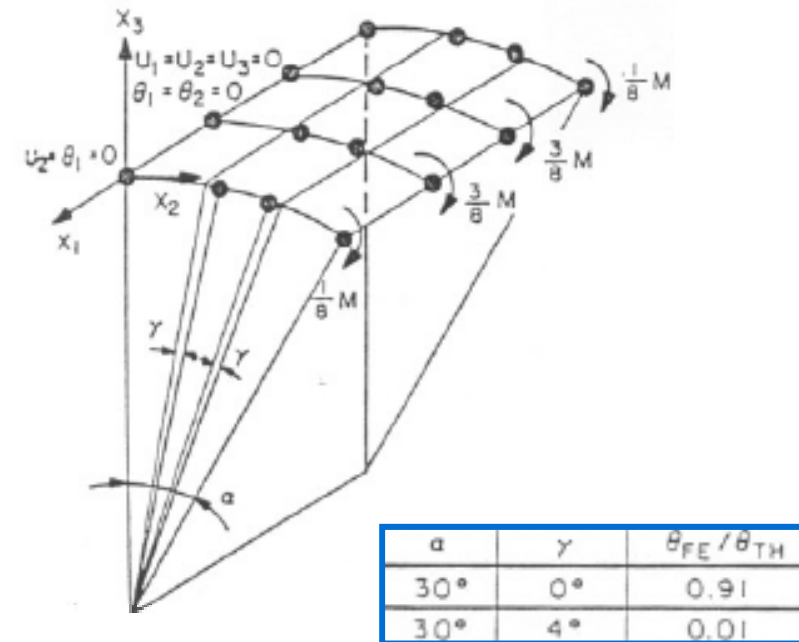


The Locking Problem

Curved cantilever



$3 \times 10^\circ$ elements $\theta_{FE} / \theta_{TH} = 0.61$
 $6 \times 5^\circ$ elements $\theta_{FE} / \theta_{TH} = 0.96$



The Figures are not to scale

The MITC4 Element

The MITC4 (mixed interpolation of tensorial components, four nodes) is a general shell element with the following features:

- It can be used in non-flat geometries (it is a shell element rather than a plate element)
- It can be used in general nonlinear analyses (material nonlinear analyses and geometrical nonlinear analyses but small strains)
- It does not lock and it does not present spurious rigid body modes.
- It can be used for thin and moderately thick shells.

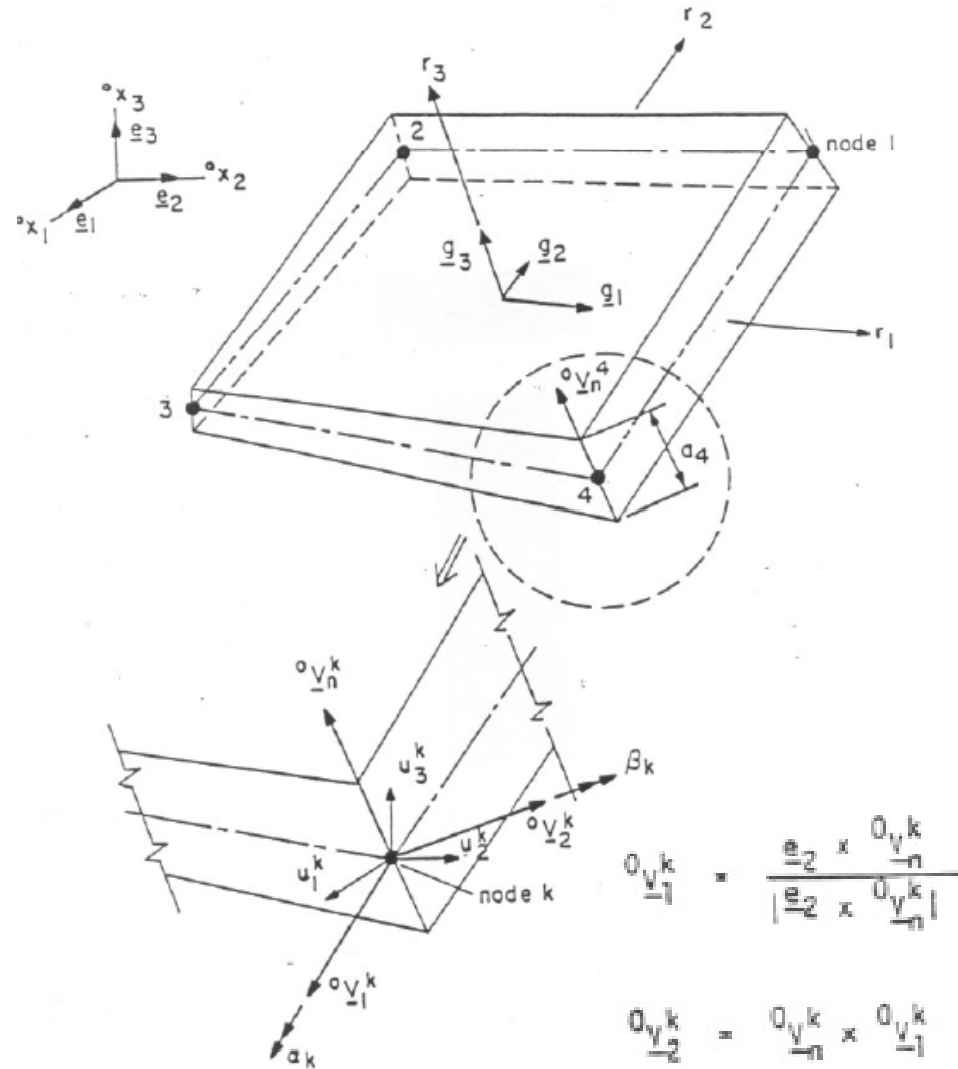
The MITC4 Element

Four node shell element

$$\underline{g}_i = \frac{\partial \underline{x}}{\partial r_i}$$

$${}^l x_i = h_k {}^l x_i^k + \frac{r_3}{2} h_k a_k {}^l V_{ni}^k$$

$l = 0$ undeformed configuration
 $l = t$ deformed configuration



The MITC4 Element

$$\underline{u}_i = h_k \underline{u}_i^k + \frac{r_3}{2} h_k a_k \left(-\alpha_k {}^l V_{2i}^k + \beta_k {}^l V_{1i}^k \right)$$

In the natural coordinate system, the strain tensor is:

$$\underline{\underline{\varepsilon}} = \tilde{\varepsilon}_{ij} \underline{g}_i \underline{g}_j$$

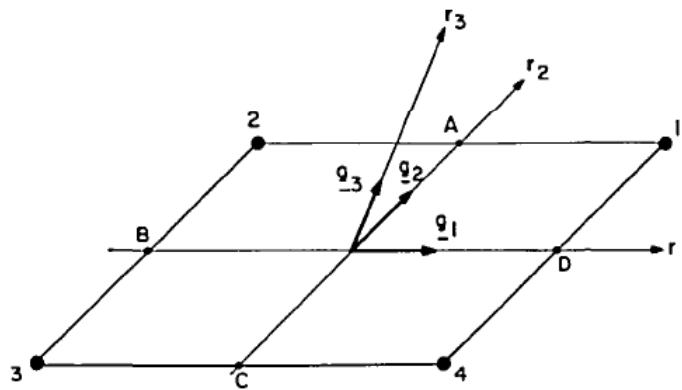
where $\tilde{\varepsilon}_{ij}$ are the strain tensor components and \underline{g}_i the base vectors

The MITC4 Element

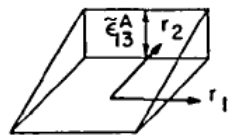
- The usual A-I-Z interpolations for the displacements/rotations, Eqn. (3)
- The “in-layer” strains ($\tilde{\varepsilon}_{11}$, $\tilde{\varepsilon}_{22}$ and $\tilde{\varepsilon}_{33}$) are directly calculated from the displacement interpolation using the kinematic relations.
- The transverse shear strains $\tilde{\varepsilon}_{13}$ and $\tilde{\varepsilon}_{33}$ are interpolated using the interpolations shown in Fig. 2

The MITC4 Element

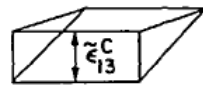
Interpolation functions for the transverse shear strains



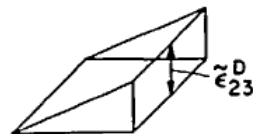
$$\tilde{\varepsilon}_{13} = \frac{1}{2} (1+r_2) \tilde{\varepsilon}_{13}|_A^{DI} + \frac{1}{2} (1-r_2) \tilde{\varepsilon}_{13}|_C^{DI}$$



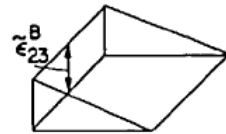
$\tilde{\varepsilon}_{13}$ interpolation



$$\tilde{\varepsilon}_{23} = \frac{1}{2} (1+r_1) \tilde{\varepsilon}_{23}|_D^{DI} + \frac{1}{2} (1-r_1) \tilde{\varepsilon}_{23}|_B^{DI}$$



$\tilde{\varepsilon}_{23}$ interpolation



MITC4 - TLF

$$\int_{\overset{0}{V}} {}^{t+\Delta t} \overset{0}{\tilde{S}}_{ij} \delta {}^{t+\Delta t} \overset{0}{\tilde{e}}_{ij} dV = {}^{t+\Delta t} \mathcal{R}$$

$${}^{t+\Delta t} \overset{0}{\tilde{S}}_{ij} = {}^t \overset{0}{\tilde{S}}_{ij} + {}^0 \tilde{S}_{ij}$$

$${}^{t+\Delta t} \overset{0}{\tilde{e}}_{ij} = {}^t \overset{0}{\tilde{e}}_{ij} + {}^0 \tilde{e}_{ij}$$

$${}^0 \tilde{e}_{ij} = {}^0 \tilde{e}_{ij} + {}^0 \tilde{\eta}_{ij}$$

$$\int_{\overset{0}{V}} {}^0 \tilde{C}_{ijkl} {}^0 \tilde{e}_{kl} \delta {}^0 \tilde{e}_{ij} dV + \int_{\overset{0}{V}} {}^t \overset{0}{\tilde{S}}_{ij} \delta {}^0 \tilde{\eta}_{ij} dV = {}^{t+\Delta t} \mathcal{R} - \int_{\overset{0}{V}} {}^t \overset{0}{\tilde{S}}_{ij} \delta {}^0 \tilde{e}_{ij} dV$$

MITC4 - TLF

$${}^0\tilde{e}_{ii} = h_{k,i} {}^t\mathbf{g}_i \cdot \mathbf{u}_k + \frac{r_3}{2} a_k h_{k,i} (-\alpha_k {}^t\mathbf{g}_i \cdot {}^t\mathbf{V}_2^k + \beta_k {}^t\mathbf{g}_i \cdot {}^t\mathbf{V}_1^k)$$

($i = 1, 2$)

$${}^0\tilde{\eta}_{ii} = \frac{1}{2} h_{k,i} h_{p,i} \mathbf{u}_k \cdot \mathbf{u}_p + \frac{r_3}{2} h_{k,i} h_{p,i} a_p (-\alpha_p {}^t\mathbf{V}_2^p \cdot \mathbf{u}_k + \beta_p {}^t\mathbf{V}_1^p \cdot \mathbf{u}_k) + \frac{(r_3)^2}{8} h_{k,i} h_{p,i} a_k a_p (-\alpha_k {}^t\mathbf{V}_2^k + \beta_k {}^t\mathbf{V}_1^k) \cdot (-\alpha_p {}^t\mathbf{V}_2^p + \beta_p {}^t\mathbf{V}_1^p)$$

$${}^0\tilde{e}_{12} = \frac{1}{2} [h_{k,2} {}^t\mathbf{g}_1 \cdot \mathbf{u}_k + h_{k,1} {}^t\mathbf{g}_2 \cdot \mathbf{u}_k + \frac{r_3}{2} h_{k,2} a_k (-\alpha_k {}^t\mathbf{V}_2^k \cdot {}^t\mathbf{g}_1 + \beta_k {}^t\mathbf{V}_1^k \cdot {}^t\mathbf{g}_1) + \frac{r_3}{2} h_{k,1} a_k (-\alpha_k {}^t\mathbf{V}_2^k \cdot {}^t\mathbf{g}_2 + \beta_k {}^t\mathbf{V}_1^k \cdot {}^t\mathbf{g}_2)]$$

$${}^0\tilde{\eta}_{12} = \frac{1}{2} [h_{k,1} h_{p,2} \mathbf{u}_k \cdot \mathbf{u}_p + \frac{r_3}{2} h_{k,1} h_{p,2} a_p (-\alpha_p {}^t\mathbf{V}_2^p \cdot \mathbf{u}_k + \beta_p {}^t\mathbf{V}_1^p \cdot \mathbf{u}_k) + \frac{r_3}{2} h_{k,1} h_{p,2} a_k (-\alpha_k {}^t\mathbf{V}_2^k \cdot \mathbf{u}_p + \beta_k {}^t\mathbf{V}_1^k \cdot \mathbf{u}_p) + \frac{(r_3)^2}{4} h_{k,1} h_{p,2} a_k a_p (-\alpha_k {}^t\mathbf{V}_2^k + \beta_k {}^t\mathbf{V}_1^k) \cdot (-\alpha_p {}^t\mathbf{V}_2^p + \beta_p {}^t\mathbf{V}_1^p)]$$

MITC4 - TLF

$${}^0\tilde{e}_{13} = \frac{1}{8}(1+r_2)[{}^t g_{3i}^A(u_i^1 - u_i^2) + \frac{1}{2}{}^t g_{1i}^A(-\alpha_1 a_1 {}^t V_{2i}^1 + \beta_1 a_1 {}^t V_{1i}^1 - \alpha_2 a_2 {}^t V_{2i}^2 + \beta_2 a_2 {}^t V_{1i}^2)] +$$

$$\frac{1}{8}(1-r_2)[{}^t g_{3i}^C(u_i^4 - u_i^3) + \frac{1}{2}{}^t g_{1i}^C(-\alpha_4 a_4 {}^t V_{2i}^4 + \beta_4 a_4 {}^t V_{1i}^4 - \alpha_3 a_3 {}^t V_{2i}^3 + \beta_3 a_3 {}^t V_{1i}^3)]$$

$${}^0\tilde{\eta}_{13} = \frac{1}{32}(1+r_2)[(-\alpha_1 a_1 {}^t V_{2i}^1 + \beta_1 a_1 {}^t V_{1i}^1 - \alpha_2 a_2 {}^t V_{2i}^2 + \beta_2 a_2 {}^t V_{1i}^2)(u_i^1 - u_i^2)] +$$

$$\frac{1}{32}(1-r_2)[(-\alpha_4 a_4 {}^t V_{2i}^4 + \beta_4 a_4 {}^t V_{1i}^4 - \alpha_3 a_3 {}^t V_{2i}^3 + \beta_3 a_3 {}^t V_{1i}^3)(u_i^4 - u_i^3)]$$

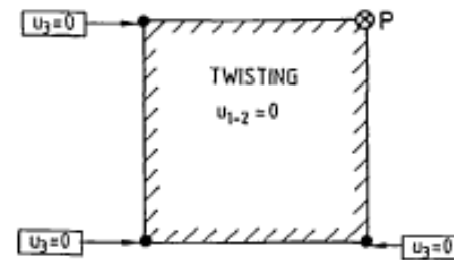
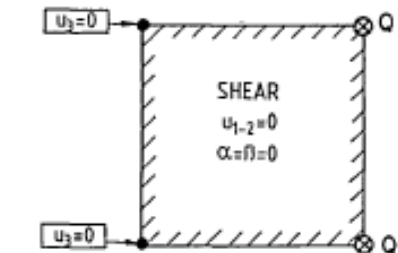
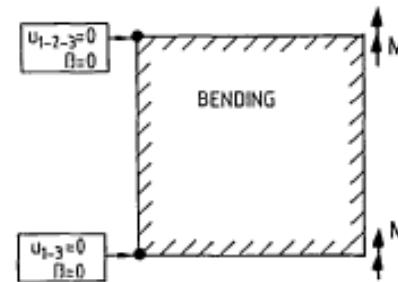
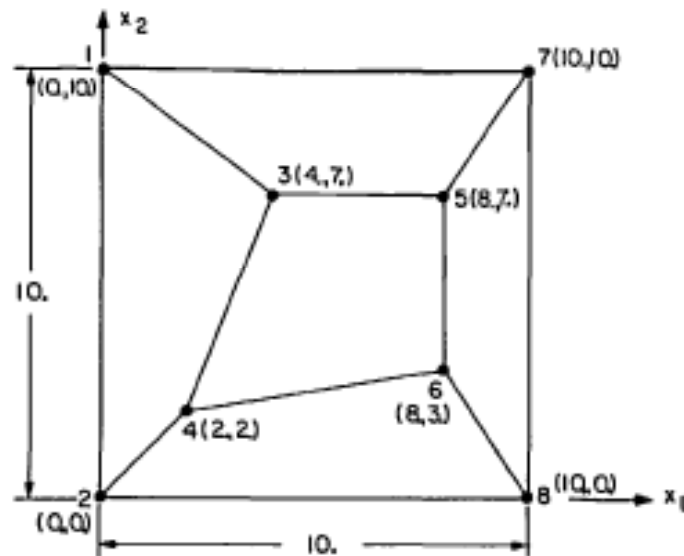
$${}^0\tilde{e}_{23} = \frac{1}{8}(1+r_1)[{}^t g_{3i}^D(u_i^1 - u_i^4) + \frac{1}{2}{}^t g_{2i}^D(-\alpha_1 a_1 {}^t V_{2i}^1 + \beta_1 a_1 {}^t V_{1i}^1 - \alpha_4 a_4 {}^t V_{2i}^4 + \beta_4 a_4 {}^t V_{1i}^4)] +$$

$$\frac{1}{8}(1-r_1)[{}^t g_{3i}^B(u_i^2 - u_i^3) + \frac{1}{2}{}^t g_{2i}^B(-\alpha_2 a_2 {}^t V_{2i}^2 + \beta_2 a_2 {}^t V_{1i}^2 - \alpha_3 a_3 {}^t V_{2i}^3 + \beta_3 a_3 {}^t V_{1i}^3)]$$

$${}^0\tilde{\eta}_{23} = \frac{1}{32}(1+r_1)[(-\alpha_1 a_1 {}^t V_{2i}^1 + \beta_1 a_1 {}^t V_{1i}^1 - \alpha_4 a_4 {}^t V_{2i}^4 + \beta_4 a_4 {}^t V_{1i}^4)(u_i^1 - u_i^4)] +$$

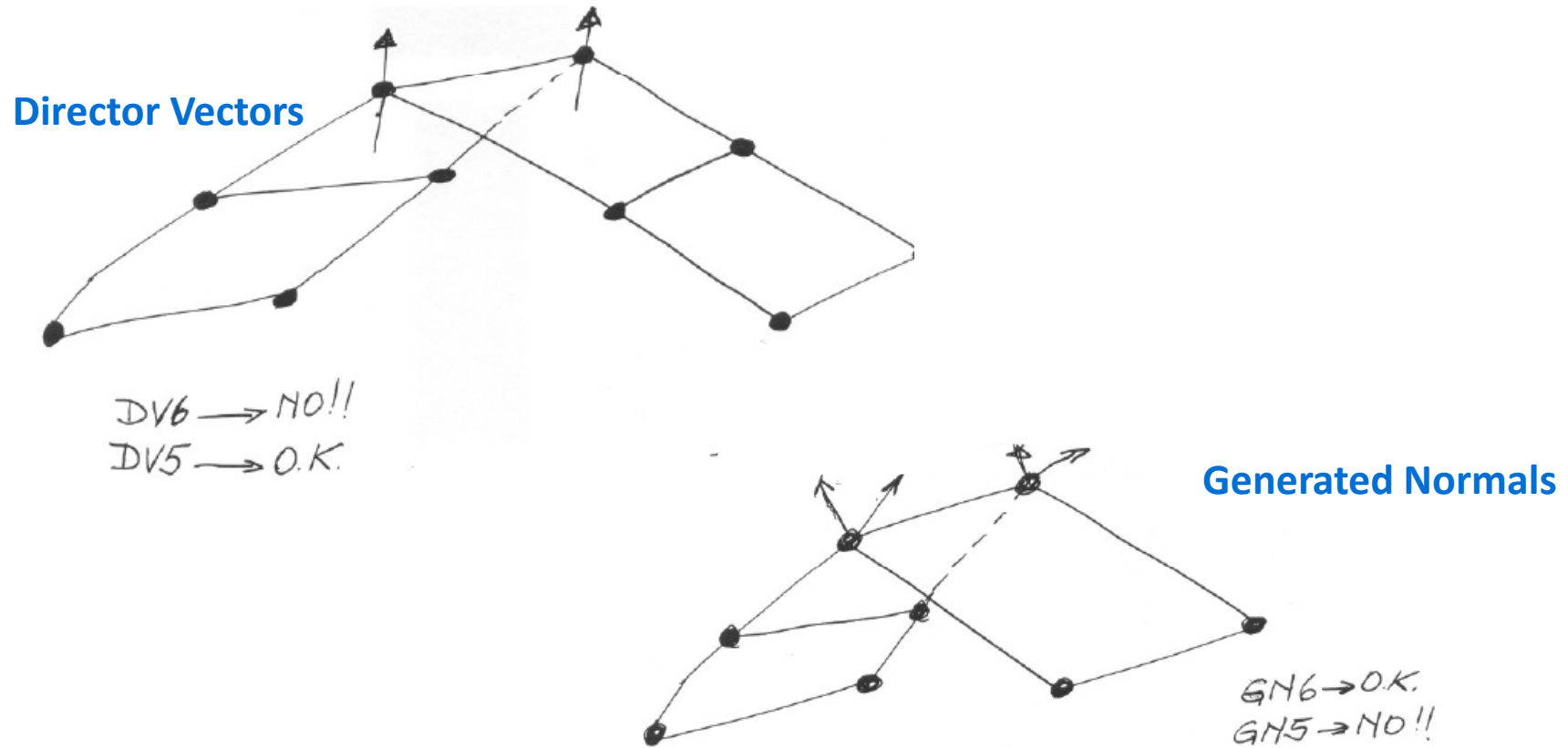
$$\frac{1}{32}(1-r_1)[(-\alpha_2 a_2 {}^t V_{2i}^2 + \beta_2 a_2 {}^t V_{1i}^2 - \alpha_3 a_3 {}^t V_{2i}^3 + \beta_3 a_3 {}^t V_{1i}^3)(u_i^2 - u_i^3)]$$

MITC4 - The Patch Test



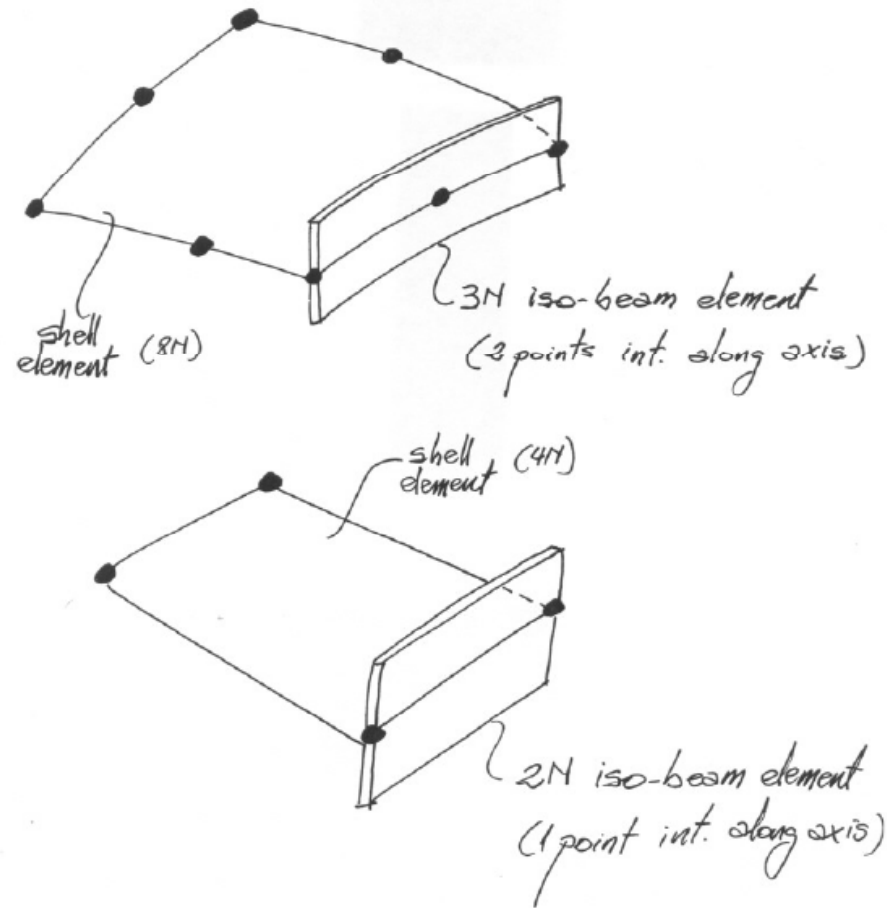
MITC4 - Modeling Details

Shell Intersections



MITC4 - Modeling Details

Modelling of stiffened shells



MITC4 - Finite Rotations

$${}^{t+\Delta t}\underline{\mathbf{V}}_n^k = {}^t{}_{t+\Delta t}\underline{\underline{\mathbf{R}}}^k \cdot {}^t\underline{\mathbf{V}}_n^k$$

$$[{}^t{}_{t+\Delta t}R^k] = [I_3] + \frac{\sin(\theta^k)}{\theta^k} [\Theta^k] + \frac{1}{2} \left[\frac{\sin(\theta^k/2)}{(\theta^k/2)} \right]^2 [\Theta^k]^2$$

In the above $\theta^k = [(\theta_1^k)^2 + (\theta_2^k)^2]^{\frac{1}{2}}$

$$[\Theta^k] = \begin{bmatrix} 0 & 0 & \theta_2^k \\ 0 & 0 & -\theta_1^k \\ -\theta_2^k & \theta_1^k & 0 \end{bmatrix}$$

- For infinitesimal incremental rotations θ_1^k and θ_2^k are independent infinitesimal rotations around ${}^t\underline{\mathbf{V}}_1^k$ and ${}^t\underline{\mathbf{V}}_2^k$ respectively.
- For finite incremental rotations θ_1^k and θ_2^k *are not independent rotations*, they are the two variables that define the rotation tensor.
- As in the infinitesimal rotations case we only have 5 d.o.f. / node.

Finite Rotations - Linearization

$$[{}^t_{t+\Delta t}R^k] = [I_3] + [\Theta^k] + \frac{1}{2!} [\Theta^k]^2 + \dots \quad (33)$$

Using the above in Eqn. (31) we get

$${}^{t+\Delta t}\underline{\mathbf{V}}_n^k - {}^t\underline{\mathbf{V}}_n^k = \underline{\theta}^k \times {}^t\underline{\mathbf{V}}_n^k + \frac{1}{2} \underline{\theta}^k \times (\underline{\theta}^k \times {}^t\underline{\mathbf{V}}_n^k) + h.o.t. \quad (34.a)$$

where we defined

$$\underline{\theta}^k = \theta_1^k {}^t\underline{\mathbf{V}}_1^k + \theta_2^k {}^t\underline{\mathbf{V}}_2^k \quad (34.b)$$

$${}^{t+\Delta t}\underline{\mathbf{V}}_n^k - {}^t\underline{\mathbf{V}}_n^k = (\theta_2^k {}^t\underline{\mathbf{V}}_1^k - \theta_1^k {}^t\underline{\mathbf{V}}_2^k) - \frac{1}{2} [(\theta_1^k)^2 + (\theta_2^k)^2] {}^t\underline{\mathbf{V}}_n^k + h.o.t.$$

$$\underline{\mathbf{u}} = h_k \underline{\mathbf{u}}_k + \frac{r_3}{2} h_k {}^t a_k (-\theta_1^k {}^t\underline{\mathbf{V}}_2^k + \theta_2^k {}^t\underline{\mathbf{V}}_1^k) - \frac{r_3}{4} h_k {}^t a_k [(\theta_1^k)^2 + (\theta_2^k)^2] {}^t\underline{\mathbf{V}}_n^k + h.o.t.$$

Finite Rotations - Linearization

$$\underline{\mathbf{u}} = \underline{\mathbf{u}}_I + \underline{\mathbf{u}}_{II} + h.o.t.$$

$$\underline{\mathbf{u}}_I = h_k \underline{\mathbf{u}}_k + \frac{r_3}{2} h_k {}^t a_k \left(-\theta_1^k {}^t \underline{\mathbf{V}}_2^k + \theta_2^k {}^t \underline{\mathbf{V}}_1^k \right)$$

$$\underline{\mathbf{u}}_{II} = -\frac{r_3}{4} h_k {}^t a_k \left[(\theta_1^k)^2 + (\theta_2^k)^2 \right] {}^t \underline{\mathbf{V}}_n^k$$

$${}^{t+\Delta t} \underline{\tilde{\mathbf{g}}}_i = {}^t \underline{\tilde{\mathbf{g}}}_i + \frac{\partial \underline{\mathbf{u}}}{\partial r_i} \quad (i = 1, 2, 3)$$

$${}^{\circ} {}^{t+\Delta t} \tilde{\varepsilon}_{ij} = \frac{1}{2} \left[{}^{t+\Delta t} \underline{\tilde{\mathbf{g}}}_i \cdot {}^{t+\Delta t} \underline{\tilde{\mathbf{g}}}_j - {}^{\circ} \underline{\tilde{\mathbf{g}}}_i \cdot {}^{\circ} \underline{\tilde{\mathbf{g}}}_j \right]$$

$${}^{\circ} {}^{t+\Delta t} \tilde{\varepsilon}_{ij} = {}^{\circ} \tilde{\varepsilon}_{ij} + \underbrace{\frac{1}{2} \left[{}^t \underline{\tilde{\mathbf{g}}}_i \cdot \frac{\partial \underline{\mathbf{u}}}{\partial r_j} + {}^t \underline{\tilde{\mathbf{g}}}_j \cdot \frac{\partial \underline{\mathbf{u}}}{\partial r_i} + \frac{\partial \underline{\mathbf{u}}}{\partial r_i} \cdot \frac{\partial \underline{\mathbf{u}}}{\partial r_j} \right]}_{{}^{\circ} \tilde{\varepsilon}_{ij} = {}^{\circ} \tilde{\varepsilon}_{ij} + {}^{\circ} \tilde{\eta}_{ij}}$$

Finite Rotations - Linearization

$$\begin{aligned}
 {}_0\tilde{e}_{ij} &= \frac{1}{2} \left[{}^t\tilde{\mathbf{g}}_i \cdot \frac{\partial \underline{\mathbf{u}}_I}{\partial r_j} + {}^t\tilde{\mathbf{g}}_j \cdot \frac{\partial \underline{\mathbf{u}}_I}{\partial r_i} \right] \\
 {}_0\tilde{\eta}_{ij} &= \frac{1}{2} \left[{}^t\tilde{\mathbf{g}}_i \cdot \frac{\partial \underline{\mathbf{u}}_{II}}{\partial r_j} + {}^t\tilde{\mathbf{g}}_j \cdot \frac{\partial \underline{\mathbf{u}}_{II}}{\partial r_i} + \frac{\partial \underline{\mathbf{u}}_I}{\partial r_i} \cdot \frac{\partial \underline{\mathbf{u}}_I}{\partial r_j} \right]
 \end{aligned}$$

MITC4 - Finite Strains

$${}^o \underline{x}(r, s, t) = h_k(r, s) {}^o \underline{x}_k + \frac{t}{2} {}^o \underline{d} a$$

$${}^o \underline{d} = \frac{h_k(r, s) {}^o \underline{V}_n^k}{\|h_k(r, s) {}^o \underline{V}_n^k\|}$$

Gebhardt H. and Schweizerhof K. (1993), "Interpolation of curved shell geometries by low order finite elements - Errors and modifications", Int. J. Numerical Methods in Engng., vol. 36, pp.287-302.

Simo J.C. , Fox D.D. and Rifai M.S. (1989), "On a stress resultant geometrically exact shell model. Part II: The linear theory; computational aspects", Comput. Meth. Appl. Mechs. Engng., vol. 73, pp.53-92.

Introduce thickness stretching

$${}^\tau \underline{x}(r, s, t) = h_k(r, s) {}^\tau \underline{x}_k + \frac{t}{2} ({}^\tau \lambda_o + {}^\tau \lambda_1 t) {}^\tau \underline{d} a$$

$${}^\tau \underline{d} = \frac{h_k(r, s) {}^\tau \underline{V}_n^k}{\|h_k(r, s) {}^\tau \underline{V}_n^k\|}$$

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$$\underline{u} = {}^{\tau+\Delta\tau}\underline{x} - {}^{\tau}\underline{x}$$

$$\begin{aligned} \underline{u}(r, s, t) = & h_k(r, s) \underline{u}_k \\ & + \frac{t}{2} a \left({}^{\tau}\lambda_o + \Delta\lambda_o + {}^{\tau}\lambda_1 t + \Delta\lambda_1 t \right) \frac{h_k(r, s) {}^{\tau+\Delta\tau}\underline{V}_n^k}{\left\| h_k(r, s) {}^{\tau+\Delta\tau}\underline{V}_n^k \right\|} \\ & - \frac{t}{2} a \left({}^{\tau}\lambda_o + {}^{\tau}\lambda_1 t \right) \frac{h_k(r, s) {}^{\tau}\underline{V}_n^k}{\left\| h_k(r, s) {}^{\tau}\underline{V}_n^k \right\|}. \end{aligned}$$

In the above, ${}^{\tau+\Delta\tau}\lambda_o = {}^{\tau}\lambda_o + \Delta\lambda_o$ and ${}^{\tau+\Delta\tau}\lambda_1 = {}^{\tau}\lambda_1 + \Delta\lambda_1$.

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For the director vector rotations we can write [35],

$${}^{\tau+\Delta\tau}\underline{V}_n^k = {}^{\tau+\Delta\tau}\underline{R} \cdot {}^{\tau}\underline{V}_n^k$$

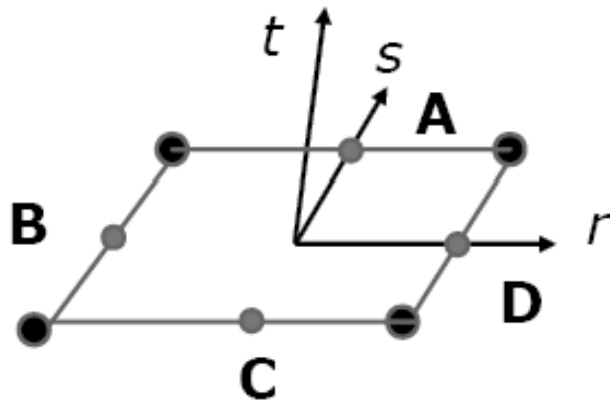
with,

$${}^{\tau+\Delta\tau}\underline{R} = \underline{I}_3 + \frac{\sin \theta_k}{\theta_k} \underline{\Theta}^k + \frac{1}{2} \left(\frac{\sin (\theta_k/2)}{(\theta_k/2)} \right)^2 (\underline{\Theta}^k)^2$$

being \underline{I}_3 is the second order unit tensor.

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$${}^{\tau}\underline{\underline{H}} = {}^{\tau}\tilde{H}_{ij} \underline{\tilde{g}}^i \underline{\tilde{g}}^j$$



$${}^{\tau}\tilde{H}_{rr} = \left[{}^{\tau}\tilde{H}_{rr} \right]^{DI}$$

$${}^{\tau}\tilde{H}_{ss} = \left[{}^{\tau}\tilde{H}_{ss} \right]^{DI}$$

$${}^{\tau}\tilde{H}_{rs} = \left[{}^{\tau}\tilde{H}_{rs} \right]^{DI}$$

$${}^{\tau}\tilde{H}_{tt} = \left[{}^{\tau}\tilde{H}_{tt} \right]^{DI}$$

$${}^{\tau}\tilde{H}_{rt} = \frac{1}{2} (1+s) \left[{}^{\tau}\tilde{H}_{rt} \right]_A^{DI} + \frac{1}{2} (1-s) \left[{}^{\tau}\tilde{H}_{rt} \right]_C^{DI}$$

$${}^{\tau}\tilde{H}_{st} = \frac{1}{2} (1+r) \left[{}^{\tau}\tilde{H}_{st} \right]_D^{DI} + \frac{1}{2} (1-r) \left[{}^{\tau}\tilde{H}_{st} \right]_B^{DI}$$

MITC4 - Finite Strains

Use 3D constitutive relations

References:

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