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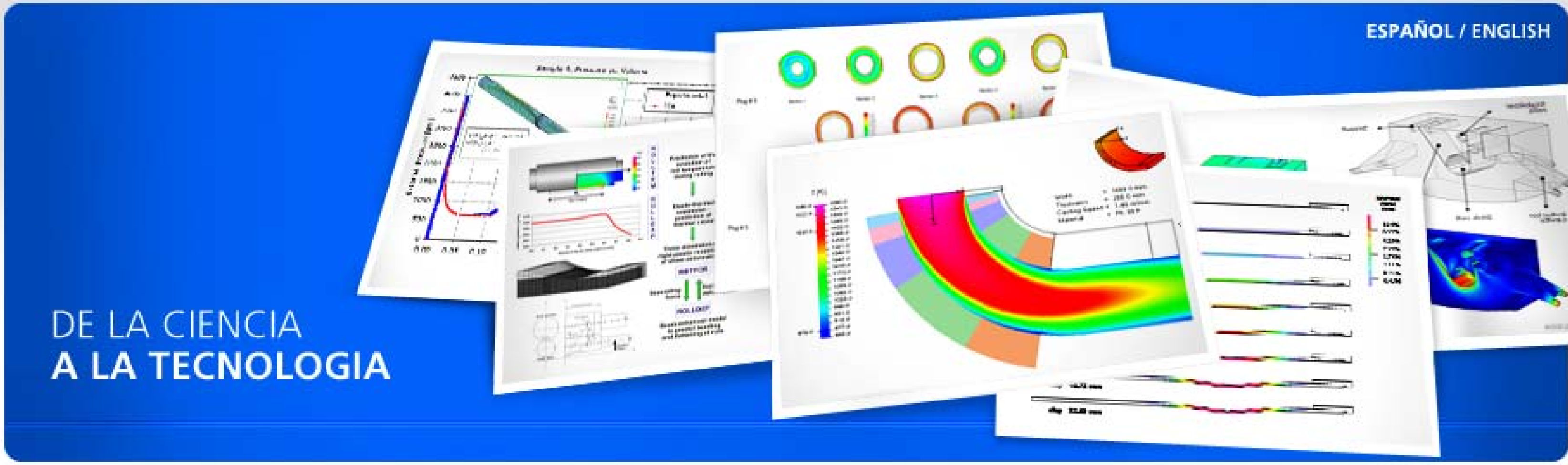

FEM in Heat Transfer
Part 3

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Part 1

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- Examples on conduction-convection heat transfer problems

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- Diffusion term
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$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p \underline{\mathbf{v}} \cdot \underline{\boldsymbol{\nabla}} T = \underline{\boldsymbol{\nabla}} \cdot \left[\underline{\underline{\mathbf{k}}}(T) \cdot \underline{\boldsymbol{\nabla}} T \right] + q_v$$

$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p \underline{\mathbf{v}} \cdot \underline{\boldsymbol{\nabla}} T = \underline{\boldsymbol{\nabla}} \cdot \left[\underline{\underline{\mathbf{k}}} \cdot \underline{\boldsymbol{\nabla}} T \right] + q_v(T)$$

$$\rho(T) C_p(T) \frac{\partial T}{\partial t} + \rho(T) C_p(T) \underline{\mathbf{v}} \cdot \underline{\boldsymbol{\nabla}} T = \underline{\boldsymbol{\nabla}} \cdot \left[\underline{\underline{\mathbf{k}}}(T) \cdot \underline{\boldsymbol{\nabla}} T \right] + q_v(T)$$

Non-linear heat transfer: thermal conductivity

Example: thermal conductivity in function of the temperature in a table

Table A.7 Gases[a]: Thermal properties

Gas	T K	k W/m K	ρ kg/m^3	c_p J/kg K
Air (82 K BP)	150	0.0158	2.355	1017
	200	0.0197	1.767	1009
	250	0.0235	1.413	1009
	260	0.0242	1.360	1009
	270	0.0249	1.311	1009
	280	0.0255	1.265	1008
	290	0.0261	1.220	1007
	300	0.0267	1.177	1005
	310	0.0274	1.141	1005
	320	0.0281	1.106	1006
	330	0.0287	1.073	1006
	340	0.0294	1.042	1007
	350	0.0300	1.012	1007
	360	0.0306	0.983	1007
	370	0.0313	0.956	1008

Non-linear heat transfer: thermal conductivity

Example: thermal conductivity in function of the temperature in a equation

Petroleum

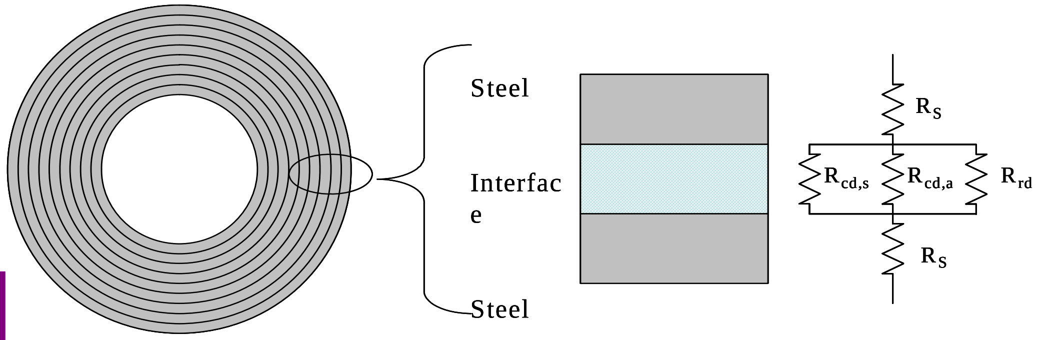
$$k = \frac{0.12}{s}\frac{1-1{,}667\,T}{10000}$$

k: [W/mK]

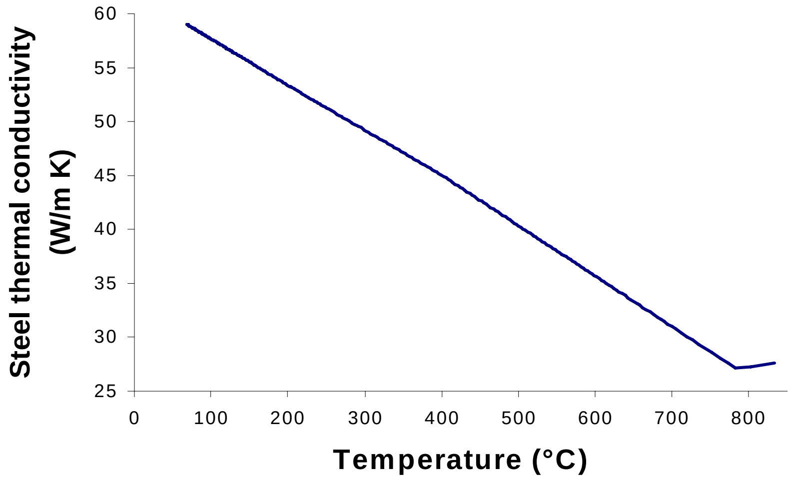
s: is the specific gravity at 60 °F

T: is the temperature [C]

Non-linear heat transfer: thermal conductivity



$$k_r = \frac{coil\ thickness}{k_S + \frac{1}{1/Rcd,s + 1/Rcd,a + 1/Rrd}}$$



ks: Thermal conductivity of steel.

Rcd,s: Interface thermal resistance of conduction through contact point.

Rcd,a: Interface thermal resistance of conduction through air.

Rrd: Interface thermal resistance of radiation through voids.

Non-linear heat transfer: thermal conductivity

$$\underline{\underline{M}} \cdot \dot{\hat{\underline{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}}(T)\right) \cdot \hat{\underline{T}} = \underline{F}$$

Picard Method

$$\left[\underline{\underline{M}} + \alpha \, \Delta t \left(\underline{\underline{N}} + \underline{\underline{K}}\left({}^{t+\Delta t}\hat{\underline{T}}^{(k-1)} \right)\right)\right] \cdot {}^{t+\Delta t}\hat{\underline{T}}^{(k)} = \alpha \, \Delta t \; {}^{t+\Delta t}\underline{F}$$

$$+ (1-\alpha)\Delta t \; {}^{t}\underline{F} - (1-\alpha)\Delta t \left(\underline{\underline{N}} + \underline{\underline{K}}\left({}^{t}\hat{\underline{T}} \right)\right) \cdot {}^{t}\hat{\underline{T}} + \underline{\underline{M}} \cdot {}^{t}\hat{\underline{T}}$$

Non-linear heat transfer: thermal conductivity

$$\underline{\underline{M}} \cdot \dot{\hat{\underline{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}}(T)\right) \cdot \hat{\underline{T}} = \underline{F}$$

Picard Method

$$\left[\underline{\underline{M}} + \alpha\,\Delta t \left(\underline{\underline{N}} + \underline{\underline{K}}\left({}^{t+\Delta t}\hat{\underline{T}}^{(k-1)}\right)\right)\right] \cdot {}^{t+\Delta t}\hat{\underline{T}}^{(k)} = \alpha\,\Delta t\;{}^{t+\Delta t}\underline{F}$$

$$+ (1-\alpha)\Delta t\;{}^{t}\underline{F} - (1-\alpha)\Delta t \left(\underline{\underline{N}} + \underline{\underline{K}}\left({}^{t}\hat{\underline{T}}\right)\right) \cdot {}^{t}\hat{\underline{T}} + \underline{\underline{M}} \cdot {}^{t}\hat{\underline{T}}$$

Non-linear heat transfer: thermal conductivity

Newton Raphson Method - Simplified

$$^{t+\Delta t}\underline{F}_{NR} = \left[\underline{\underline{M}} + \alpha\,\Delta t \left(\underline{\underline{N}} + \underline{\underline{K}}\left(^{t+\Delta t}\underline{\hat{T}}^{(k-1)} \right) \right) \right] \cdot \, ^{t+\Delta t}\underline{\hat{T}}^{(k)}$$

$$^{t+\Delta t}\underline{R}_{NR} = \alpha\,\Delta t \, ^{t+\Delta t}\underline{F} + (1-\alpha)\Delta t \, ^{t}\underline{F} - (1-\alpha)\Delta t \left(\underline{\underline{N}} + \underline{\underline{K}}\left(^{t}\underline{\hat{T}} \right) \right) \cdot \, ^{t}\underline{\hat{T}} + \underline{\underline{M}} \cdot \, ^{t}\underline{\hat{T}}$$

$$^{t+\Delta t}\underline{F}_{NR}\left(^{t+\Delta t}\underline{\hat{T}}, \, ^{t}\underline{\hat{T}} \right) = \, ^{t+\Delta t}\underline{R}_{NR}\left(^{t}\underline{\hat{T}} \right) \quad \Rightarrow \quad ^{t+\Delta t}K_{T_{ij}} = \left. \frac{\partial \, ^{t+\Delta t}F_i}{\partial \, ^{t+\Delta t}\hat{T}_j} \right|$$

Non-linear heat transfer: thermal conductivity

Newton Raphson Method - Simplified

$$ {}^{t+\Delta t}F_{NRi} = \left[M_{ij} + \alpha\,\Delta t \left(N_{ij} + K_{ij}\left({}^{t+\Delta t}\hat{\underline{T}}^{(k-1)} \right) \right) \right] {}^{t+\Delta t}\hat{T}_j^{(k)} $$

$$ \frac{\partial\, {}^{t+\Delta t}F_{NRi}}{\partial\, {}^{t+\Delta t}\hat{T}_m^{(k)}} = \left[M_{ij} + \alpha\,\Delta t \left(N_{ij} + K_{ij}\left({}^{t+\Delta t}\hat{\underline{T}}^{(k-1)} \right) \right) \right] \delta_{jm} $$

$$ {}^{t+\Delta t}\underline{\underline{K}}_T^{(k-1)} \cdot {}^{t+\Delta t}\Delta\hat{\underline{T}}^{(k)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)} $$

$$ {}^{t+\Delta t}\Delta\hat{\underline{T}}^{(k)} = {}^{t+\Delta t}\hat{\underline{T}}^{(k)} - {}^{t+\Delta t}\hat{\underline{T}}^{(k-1)} $$

Non-linear heat transfer: forces term

$$\underline{\underline{M}} \cdot \dot{\underline{\hat{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}}\right) \cdot \underline{\hat{T}} = \underline{F}$$

Volumetric therm

Heat generation due to electromagnetic forces or chemical reactions

$$q_v = C(T)T$$

$$F_i^{\,e} = \underbrace{\int_{\Omega^e} (h_i + w_i) C\left(T^{(k-1)}\right) h_j \, d\Omega \cdot \hat{T}_j^{(k)}}_{\text{on the left}} - \overbrace{\int_{\Gamma_{q^e}} h_i \, q_{n_{imp}}}^{\underline{F}}$$

$$\underline{\underline{M}} \cdot \dot{\underline{\hat{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{C}}(T)\right) \cdot \underline{\hat{T}} = \underline{F}$$

Non-linear heat transfer: volumetric term

$$\underline{\underline{M}} \cdot \dot{\underline{\hat{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{C}}(T)\right) \cdot \underline{\hat{T}} = \underline{F}$$

Picard Method

$$\Downarrow$$

$$\left[\underline{\underline{M}} + \alpha\,\Delta t\left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{C}}\left({}^{t+\Delta t}\underline{\hat{T}}^{(k-1)}\right)\right)\right] \cdot {}^{t+\Delta t}\underline{\hat{T}}^{(k)} = \alpha\,\Delta t\;{}^{t+\Delta t}\underline{F}$$

$$+ (1-\alpha)\Delta t\;{}^{t}\underline{F} - (1-\alpha)\Delta t\left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{C}}\left({}^{t}\underline{\hat{T}}\right)\right) \cdot {}^{t}\underline{\hat{T}} + \underline{\underline{M}} \cdot {}^{t}\underline{\hat{T}}$$

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Non-linear heat transfer: volumetric term

$$\underline{\underline{M}} \cdot \dot{\underline{\hat{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{C}}(T)\right) \cdot \underline{\hat{T}} = \underline{F}$$

$$\Downarrow$$

Newton Raphson Method - Simplified

$$ {}^{t+\Delta t}\underline{F}_{NR} = \left[\underline{\underline{M}} + \alpha\,\Delta t \left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{C}}\left({}^{t+\Delta t}\underline{\hat{T}}^{(k-1)} \right) \right) \right] \cdot {}^{t+\Delta t}\underline{\hat{T}}^{(k)} $$

$$ {}^{t+\Delta t}\underline{R}_{NR} = \alpha\,\Delta t\; {}^{t+\Delta t}\underline{F} + (1-\alpha)\Delta t\; {}^{t}\underline{F} $$
$$ -(1-\alpha)\Delta t \left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{C}}\left({}^{t}\underline{\hat{T}} \right) \right) \cdot {}^{t}\underline{\hat{T}} + \underline{\underline{M}} \cdot {}^{t}\underline{\hat{T}} $$

Non-linear heat transfer: volumetric term

Newton Raphson Method - Simplified

$$ {}^{t+\Delta t}F_{NR\,i} = \left[M_{ij} + \alpha\,\Delta t \left(N_{ij} + K_{ij} + C_{ij}\left({}^{t+\Delta t}\underline{\hat{T}}^{(k-1)} \right) \right) \right] {}^{t+\Delta t}\hat{T}_j^{(k)} $$

$$ \frac{\partial\, {}^{t+\Delta t}F_{NR\,i}}{\partial\, {}^{t+\Delta t}\hat{T}_m^{(k)}} = \left[M_{ij} + \alpha\,\Delta t \left(N_{ij} + K_{ij} + C_{ij}\left({}^{t+\Delta t}\underline{\hat{T}}^{(k-1)} \right) \right) \right] \delta_{jm} $$

$$ {}^{t+\Delta t}\underline{\underline{K}}_T^{(k-1)} \cdot {}^{t+\Delta t}\Delta\underline{\hat{T}}^{(k)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)} $$

$$ {}^{t+\Delta t}\Delta\underline{\hat{T}}^{(k)} = {}^{t+\Delta t}\underline{\hat{T}}^{(k)} - {}^{t+\Delta t}\underline{\hat{T}}^{(k-1)} $$

Non-linear heat transfer: forces term

$$\underline{\underline{M}} \cdot \dot{\underline{\hat{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}}(T)\right) \cdot \underline{\hat{T}} = \underline{F}$$

BC therm

$$F_i^{\ e} = \int_{\Omega^e} (h_i + w_i) q_v \, d\Omega - \int_{\Gamma_{q^e}} h_i \, h(T) h_j \, d\Gamma \cdot \hat{T}_j + \int_{\Gamma_{q^e}} h_i \, h(T) T_{amb} \, d\Gamma$$

$$^{t+\Delta t}\underline{F}_{NR} = \left[\underline{\underline{M}} + \alpha \, \Delta t \left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{H}}\left({}^{t+\Delta t}T^{(k-1)}\right)\right)\right] \cdot {}^{t+\Delta t}\underline{\hat{T}}^{(k)}$$

$$^{t+\Delta t}\underline{R}_{NR} = \alpha \, \Delta t \; {}^{t+\Delta t}\underline{F} + (1-\alpha)\Delta t \; {}^{t}\underline{F}$$
$$- (1-\alpha)\Delta t \left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{H}}\left({}^{t}T\right)\right) \cdot {}^{t}\underline{\hat{T}} + \underline{\underline{M}} \cdot {}^{t}\underline{\hat{T}}$$

Non-linear heat transfer: forces term

$$\underline{\underline{M}} \cdot \dot{\underline{\hat{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}}\right) \cdot \underline{\hat{T}} = \underline{F}(T)$$

Radiation BC

$$F_i^{\,e} = \int_{\Omega^e} (h_i + w_i) q_v \, d\Omega - \int_{\Gamma_{q^e}} h_i \, \sigma \, F \, \varepsilon \left(\tilde{T}^4 - T_{med}^{\;4}\right) d\Gamma$$

$$\frac{\partial f_i^{\,e}}{\partial \hat{T}_m} = -4 \int_{\Gamma_{q^e}} h_i \, \sigma \, F \, \varepsilon \, \tilde{T}^3 \, d\Gamma$$

$$f_i^{\,e} \approx -\int_{\Gamma_{q^e}} h_i \, \sigma \, F \, \varepsilon \, \tilde{T}^{(k-1)^4} \, d\Gamma - 4 \int_{\Gamma_{q^e}} h_i \, \sigma \, F \, \varepsilon \, \tilde{T}^{(k-1)^3} \Delta T^{(k)} \, d\Gamma \approx$$

$$\approx 3\lambda \int_{\Gamma_{q^e}} h_i \, \sigma \, F \, \varepsilon \, \tilde{T}^{(k-1)^4} \, d\Gamma - (1 + 3\lambda) \int_{\Gamma_{q^e}} h_i \, \sigma \, F \, \varepsilon \, \tilde{T}^{(k-1)^3} h_j \, d\Gamma \, \hat{T}_j^{\,(k)}$$

$0 \leq \lambda \leq 1$; $\lambda = 0$ Picard Method

$\lambda = 1$ Newton Raphson Method

Non-linear heat transfer: radiation BC

$$F_i^{\;e} = \int_{\Omega^e} \left(h_i + w_i\right) q_v \, d\Omega + \int_{\Gamma_q^e} h_i \, \sigma F \varepsilon \left(T_{amb}^{\;4} - 3\lambda \tilde{T}^{(k-1)^4}\right) d\Gamma$$

$$C_{ij}^{\;(k-1)} = \left(1 + 3\lambda\right) \int_{\Gamma_q^e} h_i \, \sigma \, F \, \varepsilon \, \tilde{T}^{(k-1)^3} h_j \, d\Gamma$$

$$\underline{\underline{M}} \cdot \dot{\hat{\underline{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}} + \underline{\underline{C}}^{(k-1)}\right) \cdot \hat{\underline{T}} = \underline{F}\left(\tilde{T}^{(k-1)}\right)$$

Non-linear heat transfer: phase change

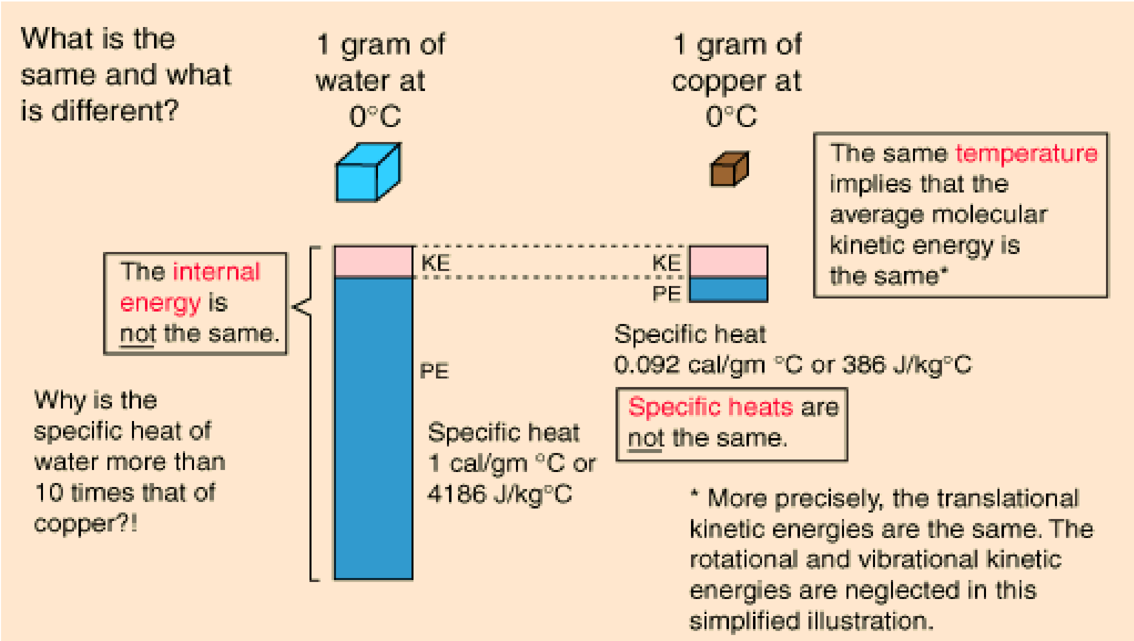
The **specific heat** is the amount of heat per unit mass required to raise the temperature by one degree Celsius.

The relationship between heat and temperature change is usually expressed in the form shown below where c is the specific heat.

The relationship does not apply if a phase change is encountered, because the heat added or removed during a phase change does not change the temperature.

The specific heat of water is 1 calorie/gram °C = 4.186 joule/gram °C which is higher than any other common substance.

Non-linear heat transfer: phase change



When the sample of water and copper are both heated by 1°C, the addition to the kinetic energy is the same, since that is what temperature measures. But to achieve this increase for water, a much larger proportional energy must be added to the potential energy portion of the internal energy. So the total energy required to increase the temperature of the water is much larger, i.e., its specific heat is much larger.

Non-linear heat transfer: phase change

There are four states, or phases, of matter:

- **Solid**
- **Liquid**
- **Gas**
- **Plasma**

We will not be discussing the plasma state here.

Change of state is when a substance changes from one state, or phase, of matter to another.

These changes of phase always occur with a **change of heat**, but the temperature does not.

Non-linear heat transfer: phase change

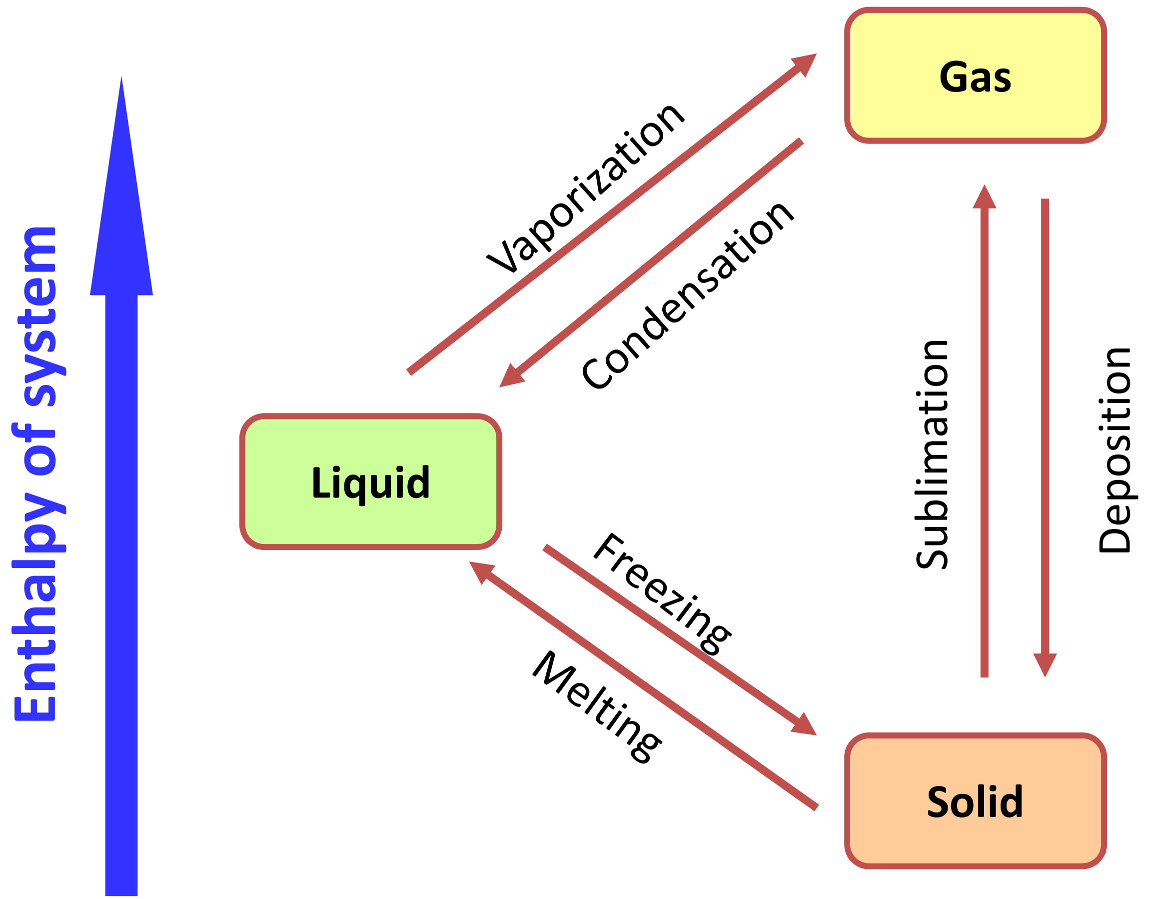
The heat exchanges present during a change in phase are changes in potential energy. These energy exchanges are *not* changes in kinetic energy.

The energy is used to break the bonds between the molecules of the substance

The change phase heat involve large amounts of energy compared to the specific heat.

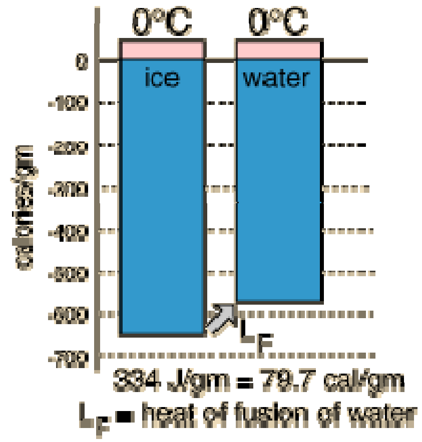
Example: ice melting into water. After the molecular bonds in the ice are broken the molecules are at a higher potential energy state, however they are not on the average moving any faster, so their average kinetic energy remains the same, and thus, their Kelvin temperature remains the same.

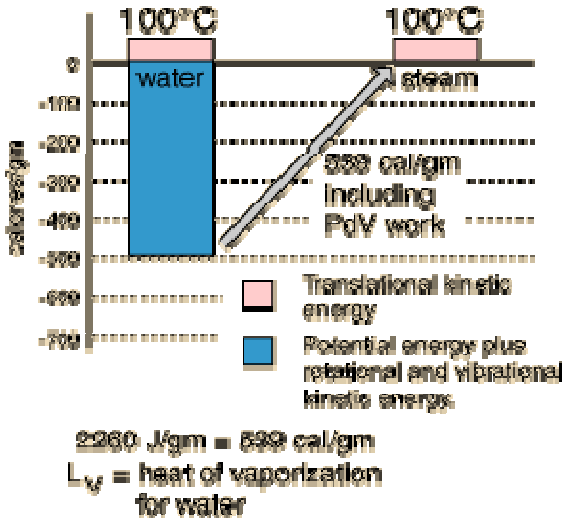
Non-linear heat transfer: phase change



Non-linear heat transfer: phase change

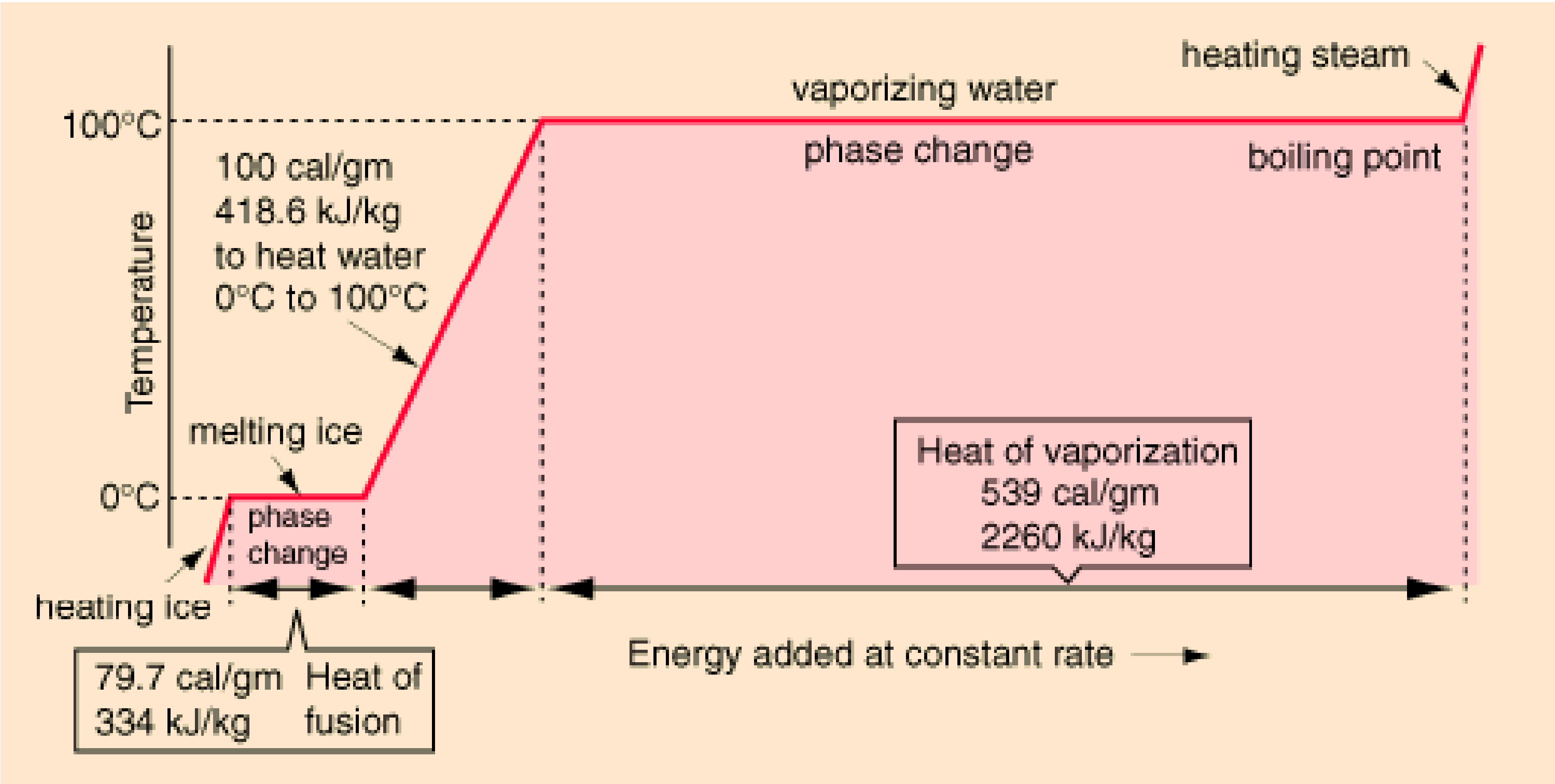
Heat of fusion: is the energy required to change a gram of a substance from the solid to the liquid state without changing its temperature.





Heat of vaporization is the energy required to change a gram of a liquid into the gaseus state.

Non-linear heat transfer: phase change



Non-linear heat transfer: phase change

Substance	Melting point K	Melting point °C	Heat of fusion (10^3 J/kg)
Helium	3.5	-269.65	5.23
Hydrogen	13.84	-259.31	58.6
Nitrogen	63.18	-209.97	25.5
Oxygen	54.36	-218.79	13.8
Ethyl alcohol	159	-114	104.2
Mercury	234	-39	11.8
Water	273.15	0.00	334
Sulfur	392	119	38.1
Lead	600.5	327.3	24.5
Antimony	903.65	630.50	165
Silver	1233.95	960.80	88.3
Gold	1336.15	1063.00	64.5
Copper	1356	1083	134

From Young, Hugh D., University Physics, 7th Ed. Table 15-4.

Non-linear heat transfer: phase change

Difussion equation:

$$\frac{\partial \hbar}{\partial t} - \underline{\nabla}\cdot\left(\underline{\underline{k}}\cdot\underline{\nabla}T\right) = Q$$

$$\frac{\partial \hbar}{\partial t} = \left.\frac{\partial \hbar}{\partial T}\right|_{X^{\phi}}\frac{\partial T}{\partial t} + \left.\frac{\partial \hbar}{\partial X^{\phi}}\right|_{T}\frac{\partial X^{\phi}}{\partial t}$$

$$-T\left.\left(\frac{\partial^2 G}{\partial T^2}\right)\right|_{P} = \rho C_p \qquad \frac{\left({}^{t+\Delta t}\hbar - {}^{t}\hbar\right)\Big|_{{}^{t+\Delta t}T}}{\Delta t} = L$$

L has a value only for the phase change temperature

$$\rho Cp\frac{\partial T}{\partial t} + L - \underline{\nabla}\cdot\left(\underline{\underline{k}}\cdot\underline{\nabla}T\right) = Q$$

Non-linear heat transfer: phase change

$$\rho Cp \frac{\partial T}{\partial t} + L - \underline{\nabla} \cdot \left(\underline{\underline{k}} \cdot \underline{\nabla} T \right) = Q$$

Latent heat formulations

Weak methods, by definitions, do not specifically account for the discontinuities in a problem.

They can not be expected to be accurate in the region near the discontinuity.

A discontinuity occurs at the phase front in freezing or melting or other phase change problems.

Non-linear heat transfer: phase change

Apparent calorific capacity formulation

$$\hbar = \int_{T_{ref}}^{T} \rho C_p(T)\,dT + \rho L\, f_l(T) = \hbar^c + \hbar^L$$

It is the local liquid volume fraction

Sensible heat

Latent heat

Using the Heaviside function for the isothermal phase change:

$$f_l = \begin{cases} 0 & T \leq T_{sol} \\ 0 < f^*(T) < 1 & T_{sol} < T < T_{liq} \\ 0 & T > T_{liq} \end{cases}$$

Non-linear heat transfer: phase change

$$\rho\, Cp_{app} = \frac{d\hbar}{dT} = \rho\, Cp + \rho\, L\, \frac{df_l}{dT}$$

$$\rho\, Cp_{app}\, \frac{\partial T}{\partial t} - \underline{\nabla} \cdot \left(\underline{\underline{k}} \cdot \underline{\nabla T} \right) = Q$$

Apparent calorific capacity method (ACCM)

$$\rho\, Cp\, \frac{\partial T}{\partial t} + \rho\, L\, \frac{\partial f_l}{\partial t} - \underline{\nabla} \cdot \left(\underline{\underline{k}} \cdot \underline{\nabla T} \right) = Q$$

Temperature based method (TBM)

Non-linear heat transfer: phase change

ACCM

$$\rho Cp_{app}\frac{\partial T}{\partial t}-\underline{\nabla}\cdot\left(\underline{\underline{k}}\cdot\underline{\nabla T}\right)=Q \longrightarrow \underline{\underline{M}}_{app}\cdot\dot{\underline{\hat{T}}}+\underline{\underline{K}}\cdot\underline{\hat{T}}=\underline{F}$$

$$M_{app\,ij}^{\;e}=\int_{\Omega^e} h_i\,\rho\,Cp_{app}\,h_j\,d\Omega$$

$$\left[\underline{\underline{M}}_{app}\left({}^{t+\Delta t}T^{(k-1)}\right)+\alpha\,\Delta t\,\underline{\underline{K}}\right]\cdot{}^{t+\Delta t}\underline{\hat{T}}^{(k)}=\alpha\,\Delta t\,{}^{t}\underline{F}+(1-\alpha)\Delta t\,{}^{t+\Delta t}\underline{F}$$
$$-(1-\alpha)\underline{\underline{K}}\cdot{}^{t}\underline{\hat{T}}+\underline{\underline{M}}_{app}\left({}^{t+\Delta t}T^{(k-1)}\right)\cdot{}^{t}\underline{\hat{T}}$$

Non-linear heat transfer: phase change

TBM

$$\rho\, Cp\, \frac{\partial T}{\partial t} + \rho\, L\, \frac{\partial f_l}{\partial t} - \underline{\nabla} \cdot \left(\underline{\underline{k}} \cdot \underline{\nabla T} \right) = Q$$

↓

$$\underline{\underline{M}} \cdot \dot{\underline{\hat{T}}} + \underline{\underline{K}} \cdot \underline{\hat{T}} + \dot{\underline{L}} = \underline{F}$$

$$\dot{L}_i^{\,e} = \int_{\Omega^e} h_i\, \rho\, L\, \frac{\partial f_l}{\partial t}\, d\Omega$$

$$\left[\underline{\underline{M}}\left({}^{t+\Delta t}T^{(k-1)} \right) + \alpha\, \Delta t\, \underline{\underline{K}} \right] \cdot {}^{t+\Delta t}\underline{\hat{T}}^{(k)} + {}^{t+\Delta t}\underline{L}^{(k)} - {}^{t}\underline{L} =$$

$$\alpha\, \Delta t\, {}^{t}\underline{F} + (1-\alpha)\Delta t\, {}^{t+\Delta t}\underline{F} - (1-\alpha)\underline{\underline{K}} \cdot {}^{t}\underline{\hat{T}} + \underline{\underline{M}}\left({}^{t+\Delta t}T^{(k-1)} \right) \cdot {}^{t}\underline{\hat{T}}$$

Non-linear heat transfer: phase change

TBM

$$\text{If} \quad {}^{t}\hat{T}_m < T_{fusion} \quad \text{and} \quad {}^{t+\Delta t}\hat{T}_m^{(k)} < T_{fusion}$$

$$\text{or}$$

$$ {}^{t}\hat{T}_m > T_{fusion} \quad \text{and} \quad {}^{t+\Delta t}\hat{T}_m^{(k)} > T_{fusion} \quad \Longrightarrow \quad {}^{t+\Delta t}\underline{L}_m^{(k)} = {}^{t}\underline{L}_m = \underline{0}$$

$$\text{If} \quad {}^{t}\hat{T}_m = T_{fusion} \quad \text{or}$$

$$ {}^{t}\hat{T}_m < T_{fusion} \quad \text{and} \quad {}^{t+\Delta t}\hat{T}_m^{(k)} < T_{fusion}$$

$$\text{or}$$

$$ {}^{t}\hat{T}_m > T_{fusion} \quad \text{and} \quad {}^{t+\Delta t}\hat{T}_m^{(k)} > T_{fusion} \quad \Longrightarrow \quad \dot{L}_m^e = \int_{\Omega^e} h_i \, \rho L \frac{\partial f_l}{\partial T} \frac{\partial T}{\partial t} \, d\Omega$$

Non-linear heat transfer: phase change

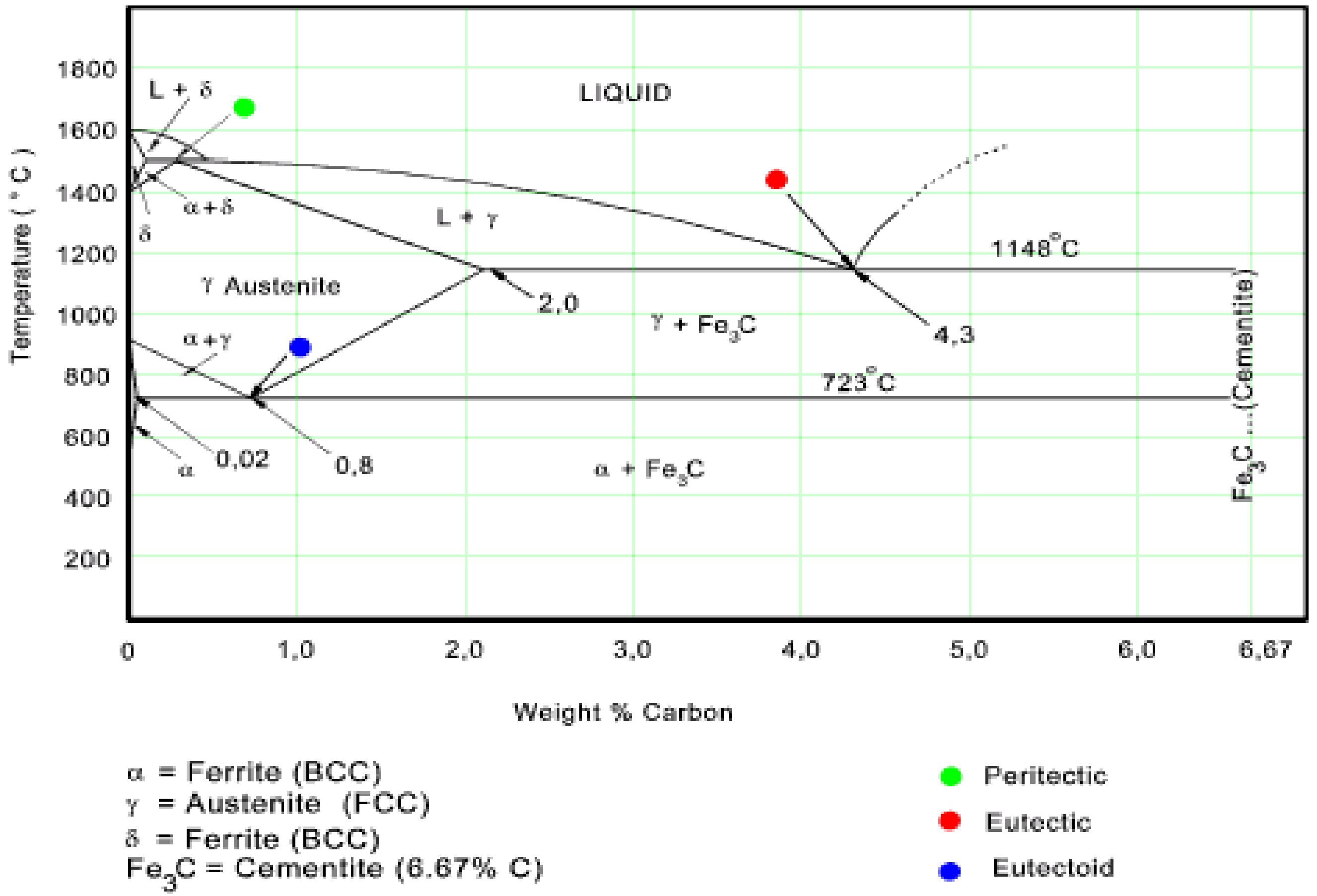
TBM

when the phase change is isothermal the $\dfrac{\partial f_l}{\partial T}$ takes the form of the dirac delta

$$ {}^{t+\Delta t}L_i^{(k)e} = \int_{\Omega^e} h_i\, \rho\, L \frac{\partial f_l}{\partial T} h_j d\Omega \;\; {}^{t+\Delta t}T_j^{(k)} \approx \int_{\Omega^e} h_i\, \rho\, L \frac{1}{\|\nabla T\|} h_j d\Omega \;\; {}^{t+\Delta t}T_j^{(k)} $$

$$ {}^{t}L_i^{(k)e} = \int_{\Omega^e} h_i\, \rho\, L \frac{\partial f_l}{\partial T} h_j d\Omega \;\; {}^{t}T_j \approx \int_{\Omega^e} h_i\, \rho\, L \frac{1_l}{\|\nabla T\|} h_j d\Omega \;\; {}^{t}T_j $$

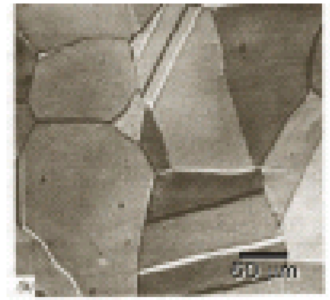
Non-linear heat transfer: phase change



Non-linear heat transfer: phase change

Solid phase

Ferrite phase



Austenite phase

α Ferrite At 0% C this is pure iron. BCC crystal structure.
The carbon atoms are located in the crystal interstices.

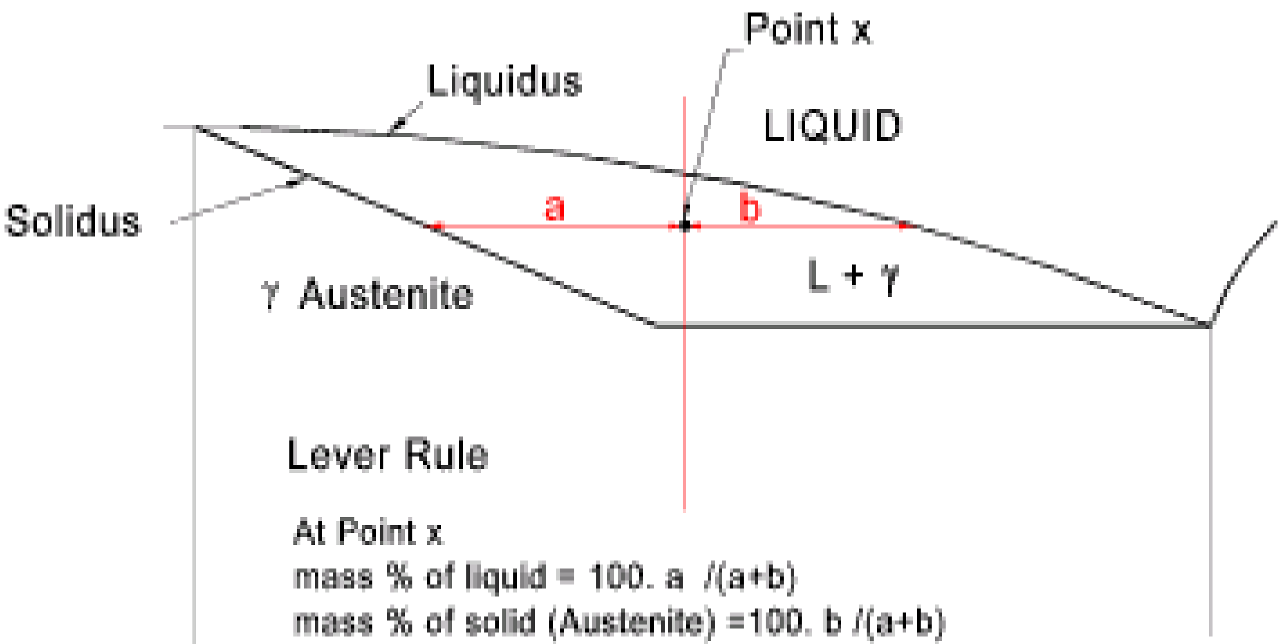
Austenite: The solid solution of carbon in γ iron is called austenite . This has a FCC crystal structure with a high solubility for carbon compared with α ferrite. The carbon atoms are dissolved interstitially. The difference in solubility between the austenite and α Ferrite is the basis for the hardening of steels

Cementite: This is an intermetallic compound. Cementite is a hard brittle compound with and orthorhombic crystal structure each unit cell has 12 Fe atoms and 4 C atoms

δ Ferrite: This is a solid solution of carbon in iron and has a BCC crystal structure. This has no real practical significance in engineering.

Non-linear heat transfer: phase change

The **lever rule** can be applied to any phase region an provides an indication of the proportions of the constituent parts at any point on the phase diagram.



Non-linear heat transfer: phase change

Phase change

$$\frac{d\mathcal{H}}{dT} = \rho_s\,\overline{c}_s \quad (for\ T \leq T_s\ ) \tag{3a}$$

$$\frac{d\mathcal{H}}{dT} = (\mathcal{H}_l - \mathcal{H}_s)\,/\,(T_l - T_s) \quad (for\ T_s < T < T_l) \tag{3b}$$

$$\frac{d\mathcal{H}}{dT} = \rho_l\,\overline{c}_l \quad (for\ T \geq T_l). \tag{3c}$$

Notice that,

- Equation (3a) is valid for $T \leq T_s$, where T_s is the *solidus* temperature. In this equation ρ_s is the solid density and $\overline{c}_s$ is the solid specific heat per unit mass (we assume it constant). Notice that the solid properties may correspond to the γ or α phases or to a state where we have a phase transformation, in this case a formula similar to Eqn. (3b) is used.
- Equation (3b) is valid for $T_s < T < T_l$, where T_l is the liquidus temperature. In this equation $\mathcal{H}_l$ and $\mathcal{H}_s$ are the enthalpies per unit volume at the *liquidus* and *solidus* lines respectively (we assume $\frac{d\mathcal{H}}{dT}$ to be constant inside the mushy zone).
- Equation (3c) is valid for $T \geq T_l$. In this equation ρ_l is the liquid density and $\overline{c}_l$ is the liquid specific heat per unit mass (we assume it constant).

Non-linear heat transfer: phase change

$$\int_v \underline{H}^T\, Cp\underline{H}.\dot{\widehat{\underline{T}}}\, dv + \int_v \underline{\nabla H}^T . k\underline{\nabla H}.\widehat{\underline{T}}\, dv + \int_v \underline{H}^T\, L\, dv$$

$$+ \int_{\Omega_q} \underline{H}^T\, Q_s\, dS + \int_{\Omega_c} \underline{H}^T\, h\, (\underline{H}.\widehat{\underline{T}} - T_{amb})\, dS + \int_{\Omega_r} \underline{H}^T\, \sigma\, F\, \epsilon\, (\widetilde{T}^4 - T^4_{medio})\, dS = 0$$

- Esquema Quasi-Newton:

$${}^{t+\Delta t}\mathbf{K}^{(i-1)}\quad \Delta\underline{T}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}$$

$${}^{t+\Delta t}\underline{T}^{(i)}_{LS} = {}^{t+\Delta t}\underline{T}^{(i-1)} + \beta^{(i)}_{LS}\, \Delta\underline{T}^{(i)}$$

- Método Mixto: $0 \leq \lambda \leq 1$

$$\underline{f}(\widetilde{T}) = \int_{\Omega_r} \underline{H}^T\, \sigma\, F\, \epsilon\, \widetilde{T}^4\, dS = (1+3\lambda) \int_{\Omega_r} \underline{H}^T\, \sigma\, \epsilon\, F\, \left({}^{t+\Delta t}\widetilde{T}^{(i-1)}\right)^3 \underline{H}\, dS\, .\, {}^{t+\Delta t}\widehat{\underline{T}}^{(i)}$$

$$-(3\lambda) \int_{\Omega_r} \underline{H}^T\, \sigma\, \epsilon\, F\, \left({}^{t+\Delta t}\widetilde{T}^{(i-1)}\right)^4 dS$$

Non-linear heat transfer: phase change

- Esquema de integración implícito $\quad {}^{t+\Delta t}\dot{\widehat{\underline{T}}}^{(i)} = \dfrac{{}^{t+\Delta t}\widehat{\underline{T}}^{(i-1)} + \Delta\underline{T}^{(i)} - {}^{t}\widehat{\underline{T}}}{\Delta t}$

resultan las siguientes matrices globales:

$$
\begin{aligned}
{}^{t+\Delta t}\mathbf{K}^{(i-1)} &= \frac{1}{\Delta t}\mathbf{K}_{cap} + \mathbf{K}_{cond} + \mathbf{K}_{conv} + (1+3\lambda)\,\mathbf{K}_{rad} \\
{}^{t+\Delta t}\underline{R} &= -\underline{R}_{Sup} \\
{}^{t+\Delta t}\underline{F}^{(i-1)} &= \frac{1}{\Delta t}\mathbf{K}_{cap}\,.\left({}^{t+\Delta t}\widehat{\underline{T}}^{(i-1)} - {}^{t}\widehat{\underline{T}}\right) + \mathbf{K}_{cond}\,.\,{}^{t+\Delta t}\widehat{\underline{T}}^{(i-1)} + \mathbf{K}_{conv}\,.\,{}^{t+\Delta t}\widehat{\underline{T}}^{(i-1)} \\
&\quad + (1+3\lambda)\,\mathbf{K}_{rad}\,.\,{}^{t+\Delta t}\widehat{\underline{T}}^{(i-1)} + \underline{F}_{lat} - \underline{F}_{conv} - \underline{F}_{rad} - (3\lambda)\,\underline{F}_{rad\_2}
\end{aligned}
$$

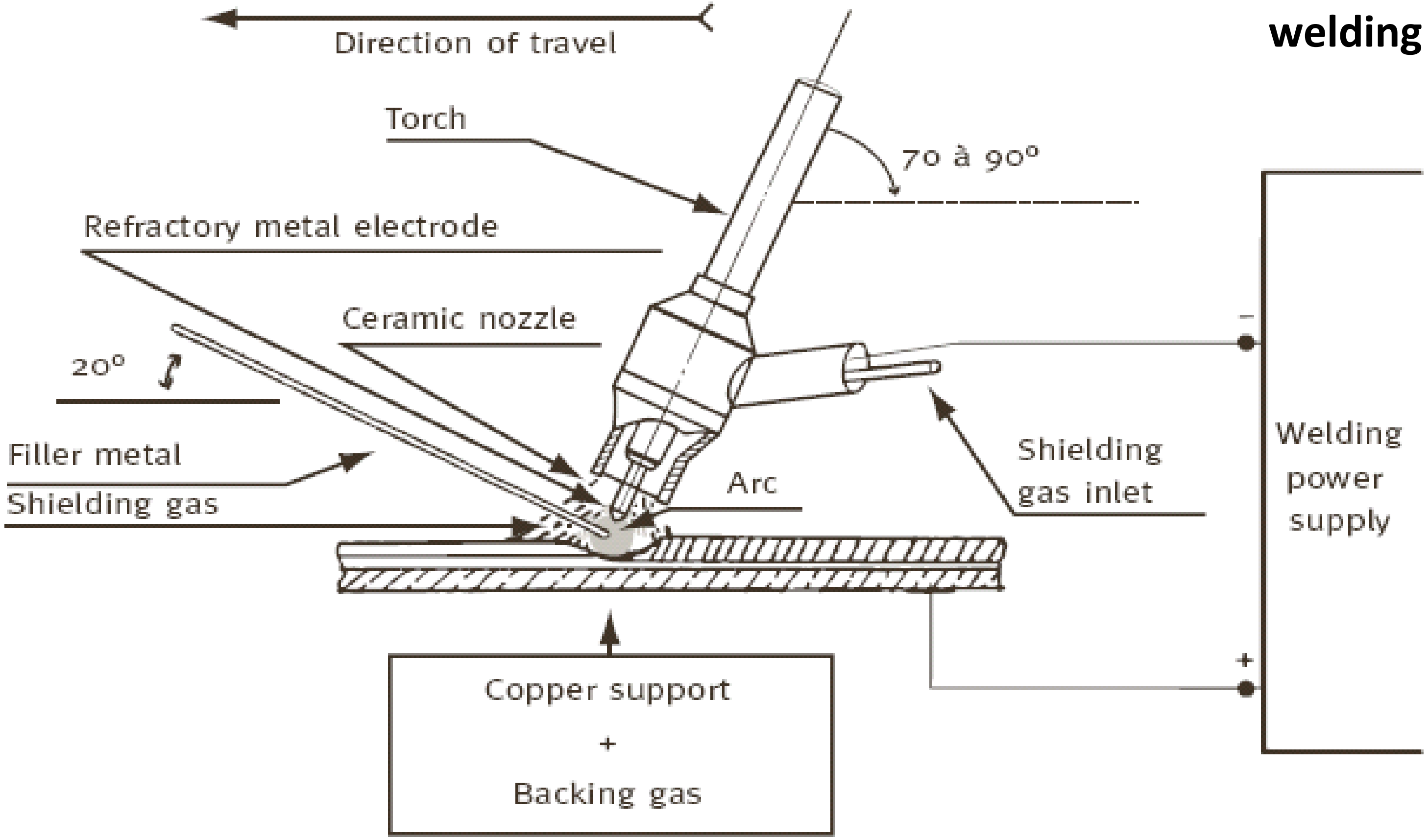
$$
\Delta\underline{T}^{(i)} = \left[{}^{t+\Delta t}\mathbf{K}^{(i-1)}\right]^{-1} \left({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}\right)
$$

Modeling of heat transfer: welding

- Arc Welding
  - TIG
  - Submerged arc welding
- Laser Welding
- Resistant Spot Welding
- Cladding
- Friction and Friction Stir Welding
- Heat treatment
- Machining
- Forming

Modeling of heat transfer: welding

Gas tungsten arc welding (GTAW)



Principle of manual gas tungsten-arc welding

Modeling of heat transfer: welding

Gas tungsten arc welding (GTAW) is also known as the TIG (Tungsten Inert Gas) or WIG (Wolfram Inert Gas).

The energy necessary for melting the metal is supplied by an electric arc struck and maintained between a tungsten (or tungsten alloy) electrode and the work piece, under an inert or slightly reducing atmosphere.

If a filler metal is employed, it is in the form of bare rods or coilec wire for automatic welding.

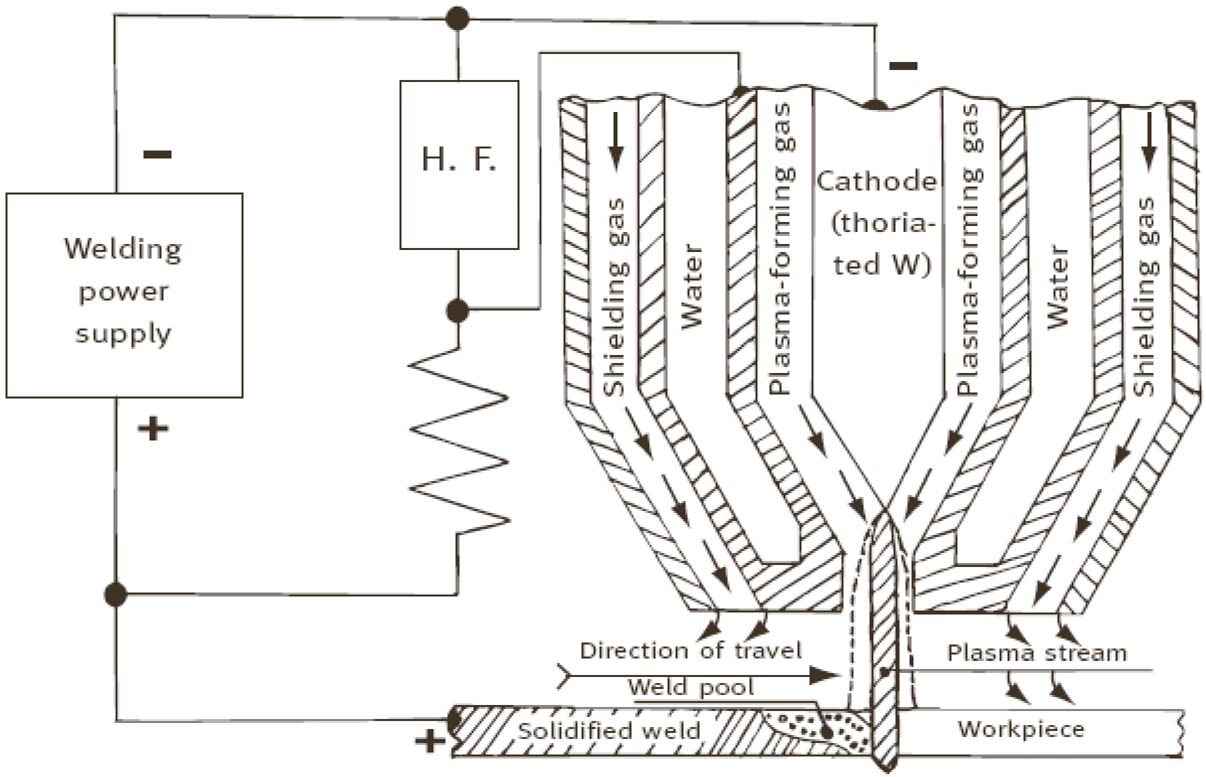
The inert gas (Ar, He, H2) flow which protects the arc zone from the ambient air, enables a very stable arc to be maintained.

Advantages:

- a concentrated heat source, leading to a narrow fusion zone
- a very stable arc and calm welding pool of small size.
- very low electrode wear

Modeling of heat transfer: welding

Plasma arc welding (PAW)



Modeling of heat transfer: welding

Plasma arc welding (PAW) is constricted by a nozzle to produce a high-energy plasma stream in which temperatures between 10000 and 20000 °C are attained.

Advantages:

- a rigid arc which enables better control of power input
- greater tolerance to variations in nozzle-workpiece distance
- a narrow heat-affected zone (HAZ)

Modeling of heat transfer: welding

Complex

Temperature distribution

Welding residual stresses and deformation

Stresses during welding

Fluid flow in arc and weld pool

Vaporization

Solidification

Metallurgical effects and transformation

Damage (creep, cracking)

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Modeling of heat transfer: welding

The numerical model includes :

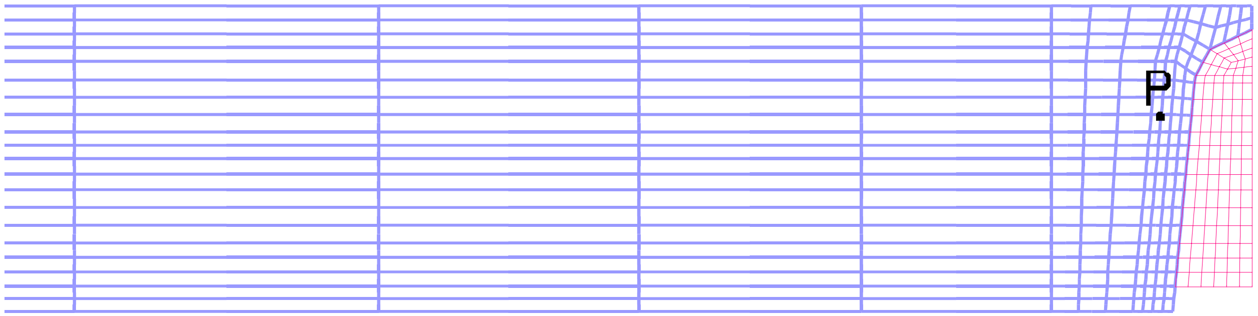
⇒ Convection and radiation heat transfer.

⇒ Temperature dependent material (specific heat and conductivity). Latent heat due to phase change.

⇒ Heat input as a function of the voltage, current intensity and wire feed speed during each pass.

⇒ Birth of elements that model the welding material.

Finite element mesh



Welding: heat flux applied on the base material surface in each pass

$$Q = \eta V I - Q_L$$

η : is the arc efficiency, *h = 0.9*

V : is the weld voltage

I : is the weld current

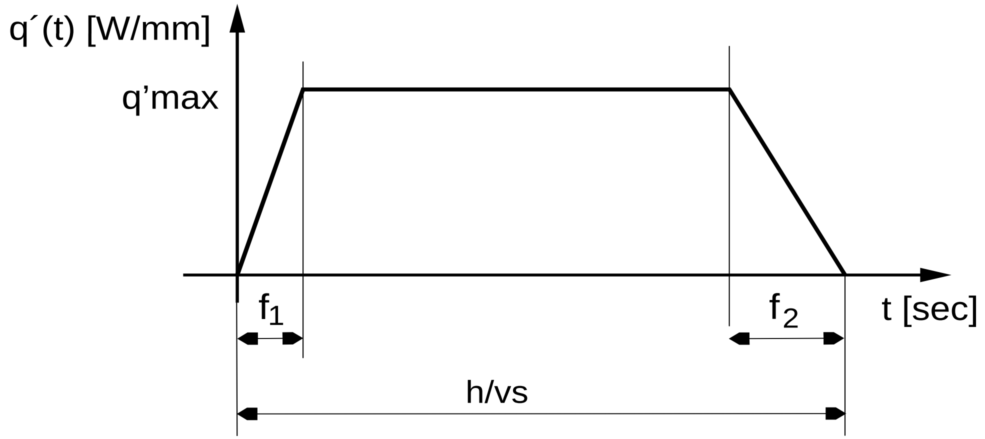
Q_L : is the power used to heat and melt the filler metal $Q_L = \left[\int_{T_{amb}}^{T_L} c\, dT + \rho\, L \right] v_e\, A_e$

T_{amb} : is the ambient temperature

T_L : is the temperature at which the filler metal is added ($T_L = 1600\ °C$)

ρ : is the density, *r = 7.5 gr/cm*^{3}

Welding: heat flux per length



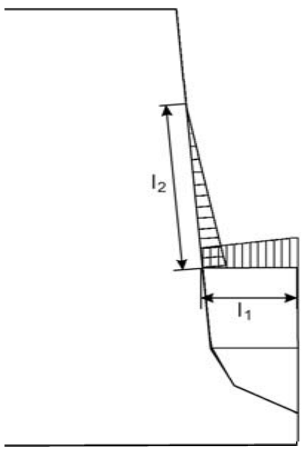
Advance direction

$$q'_{max} = \frac{40}{37}\frac{Q}{h}$$

h = 6.6 mm

v_s is the travel speed

$$f_1 = \frac{h}{20\, v_s} \qquad f_2 = \frac{h}{10\, v_s}$$



U-groove direction

q_{max} : is the maximum heat flux per area at the weld center

$$q_{max} = 2\frac{q'_{max}}{2\,l} = \frac{80}{37}\frac{Q}{2\;l\;h}$$

$$l = l_1 + l_2$$

Welding: boundary conditions

$$q_c = h\left(T - T_{amb}\right) \qquad\qquad q_r = \sigma\,\varepsilon\left(T^4 - T_{amb}{}^4\right)S_F$$

h : is the convective coefficient, $h = 5.88\ W/m^2\ °C$

s : is the Stefan Boltzmann constant, $s = 5.67\ 10^{-8}\ W/m^2\ °K^4$

e : is the surface emissivity, $e = 0.8$

S_F : is a radiation geometric factor

For the lower surface (internal of the pipe) : $T_{amb} = 170\ °C$, $S_F = 1$

For the upper surface (external of the pipe) : $T_{amb} = 170\ °C$, $S_F = 1$

For the U-groove surface: $T_{amb} = 170\ °C$, $S_F = 0.125$

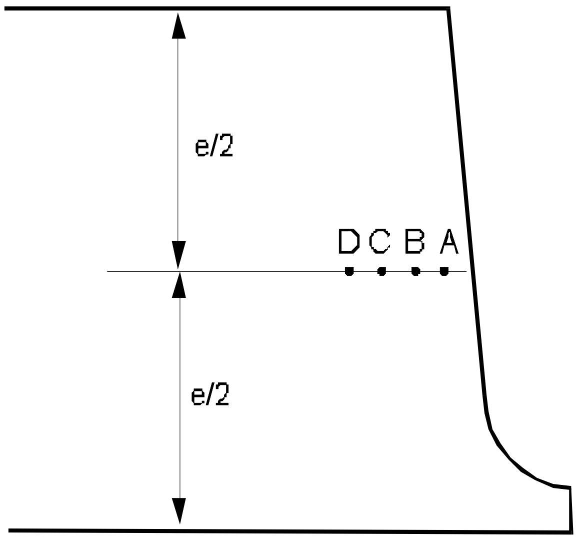
Welding example: CRC Evans Automatic test

Electrode	ER70S-G	Direct Current
Electrode diameter	0.88 mm	Positive electrode
Wire feed speed	161-319 mm/s	Oscillation
Gas	85Ar/15CO_2	Position 5G
Preheat temperature	115-150ºC	
Interpass temperature	250ºC maximum	

Welding example: Parameters

Pass	*Voltage(V)*	*Current (A)*	*Travel speed(mm/s)*	*HI (J/mm)*
Root	21.1-22.1	208-228	14.7-14.8	290-340
Hot Pass	23.1-24.9	197-205	7.5-7.7	590-670
Fill 1-2	22.9-24.5	190-214	6.7-6.8	630-780
Fill 3	22.0-23.9	191-224	6.1-9.6	430-880
Cap	22.7-25.7	149-170	4.2-6.3	540-1040

Welding example



Distance of the surface

A = 0.5 mm

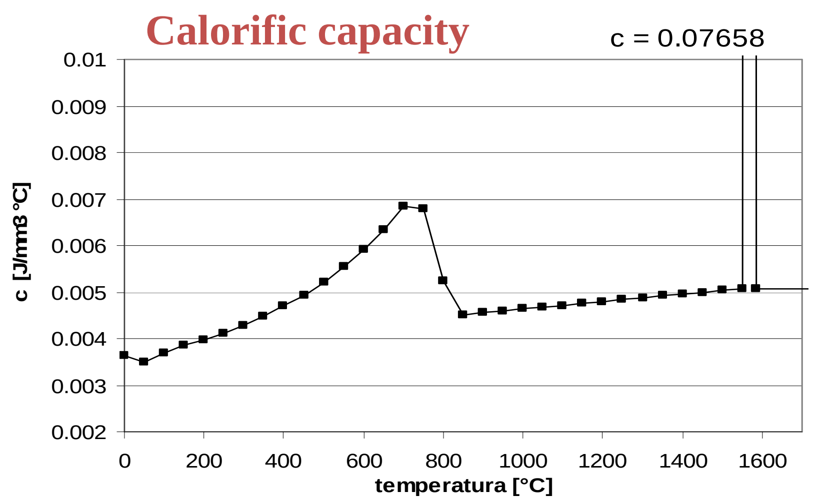
B = 1 mm

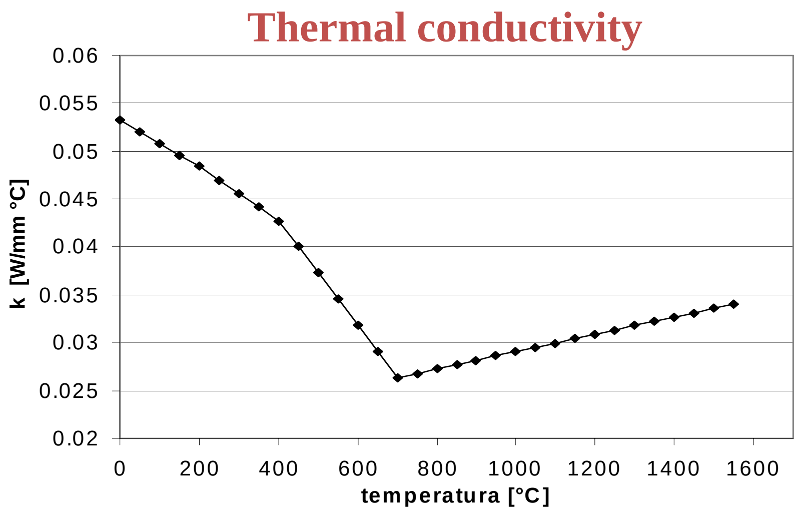
C = 1.5 mm

D = 2 mmd

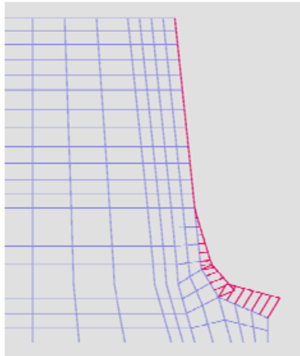
Welding: Properties in function of the chemical composition and temperatures

C = 0.110	Mn = 1.020	Si = 0.250	P = 0.012	S = 0.002
Mo = 0.080	Cr = 0.080	V = 0.044	Nb= 0.022	Ni = 0.090
Cu = 0.120	Sn = 0.006	Al = 0.026	Ti = 0.010	

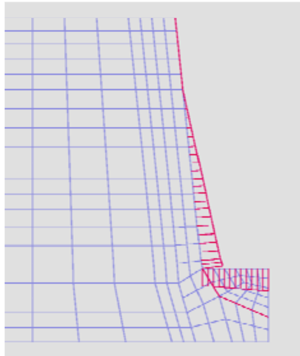




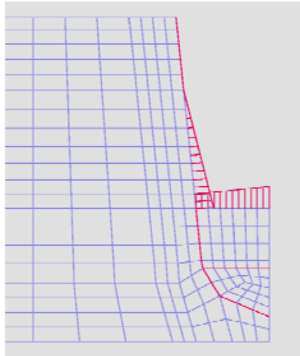
Welding: heat input



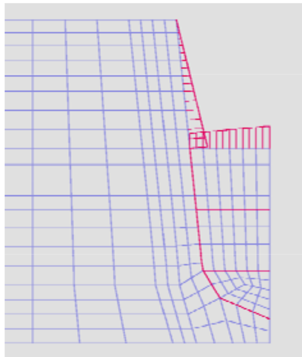
Q_{max} = 62.05 W/mm^2



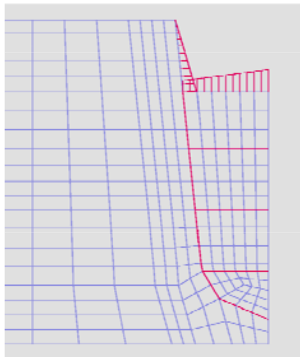
Q_{max} = 37.89 W/mm^2



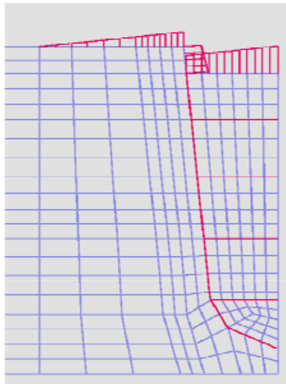
Q_{max} = 48.80 W/mm^2



Q_{max} = 44.67 W/mm^2

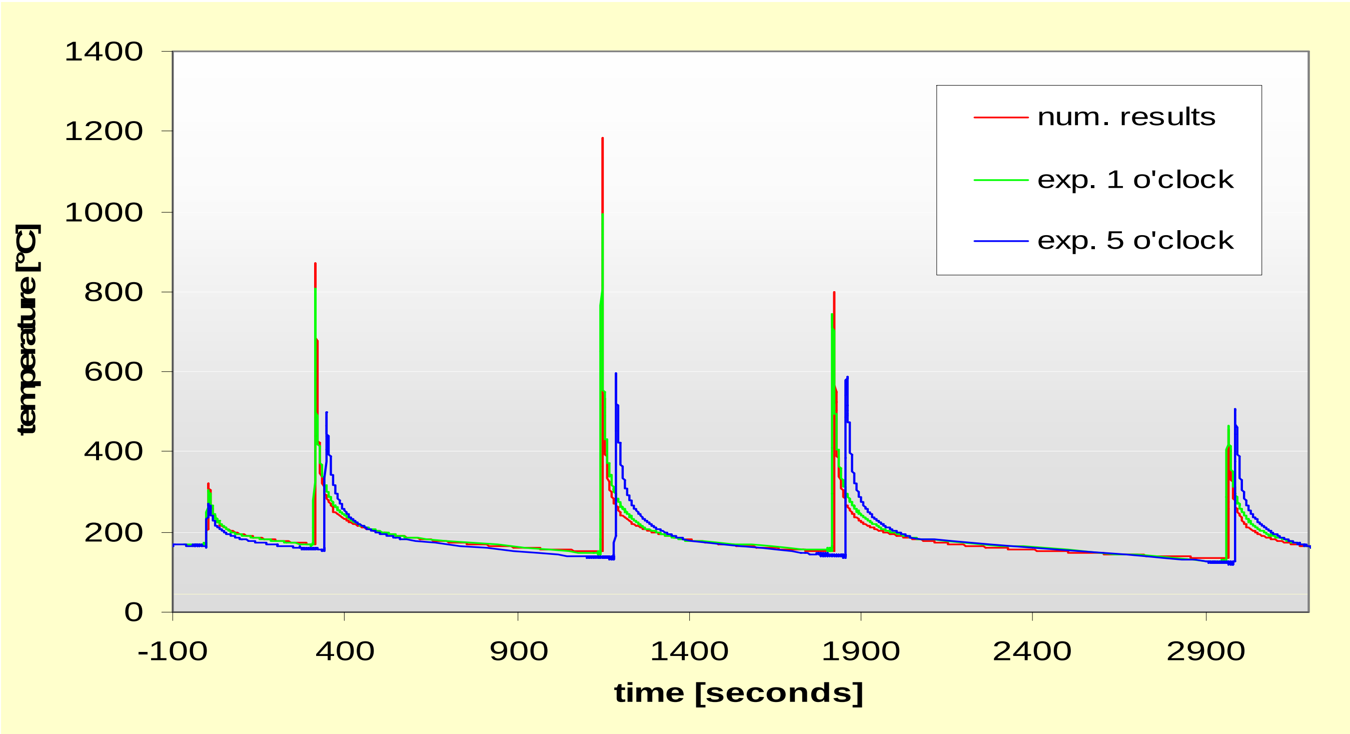


Q_{max} = 60.31 W/mm^2

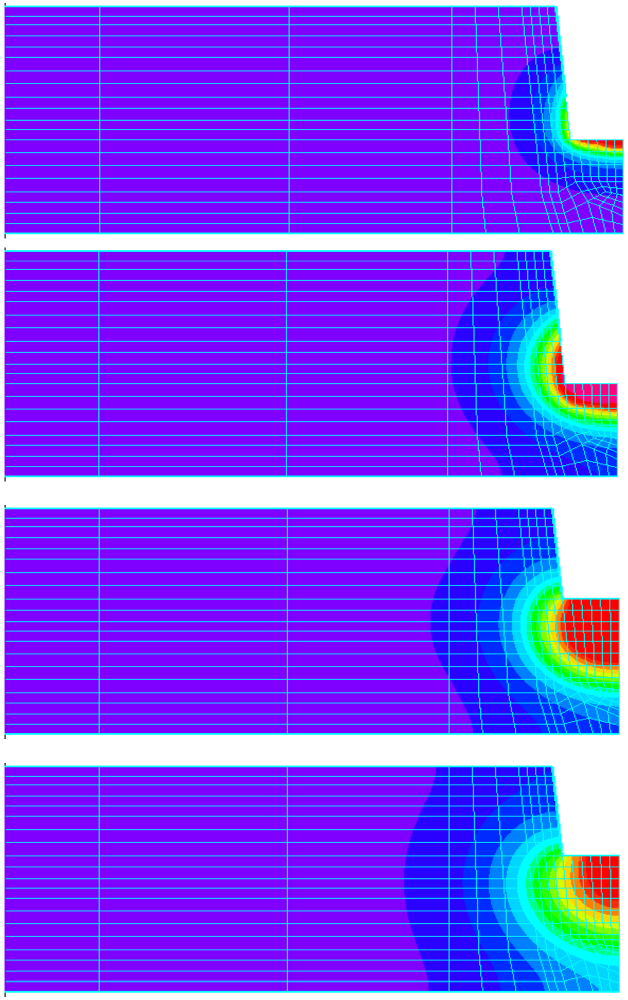


Q_{max} = 33.70 W/mm^2

Welding: Comparison between the measured temperatures and the numerically predicted ones



Results for point B (1mm of the surface) shown on the finite element mesh

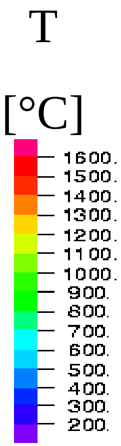


time = -0.8 s

time = -0.1 s

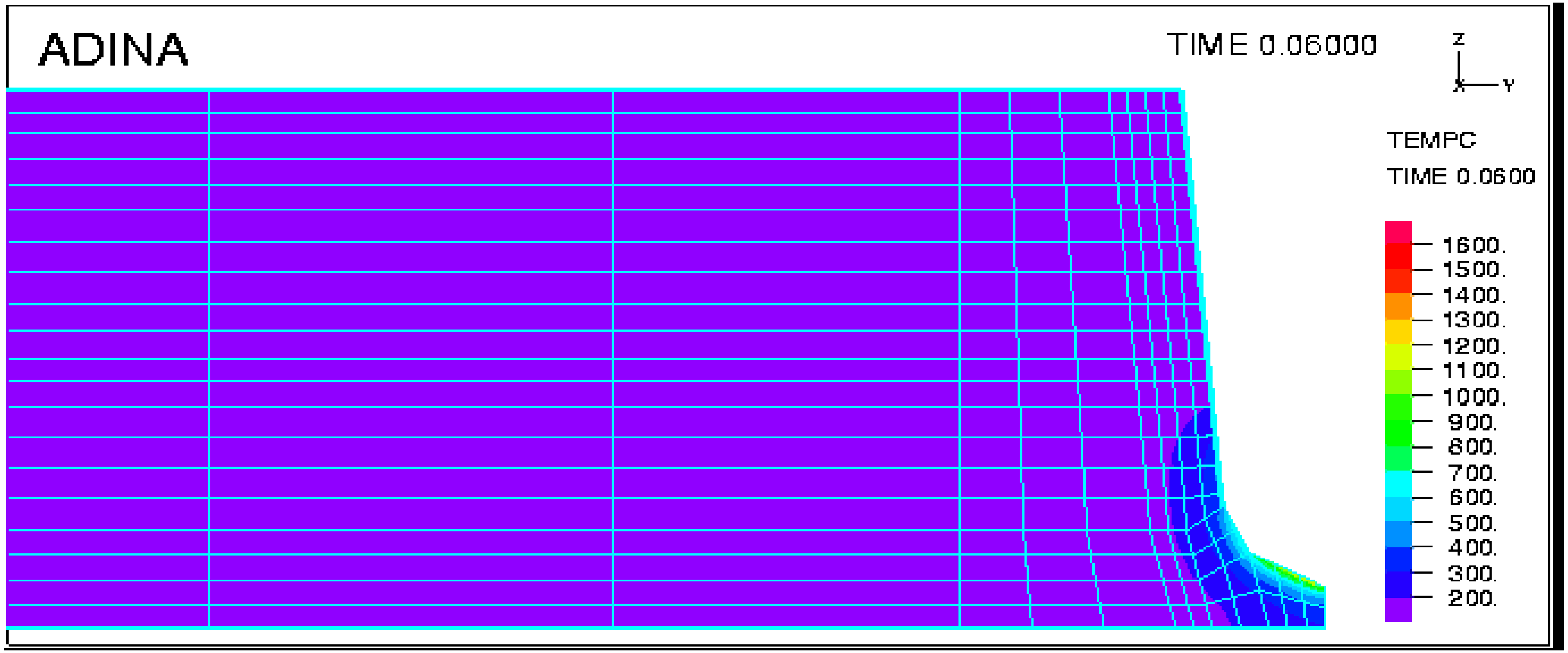
time = 0.1 s

time = 0.5 s



time = 0 indicates the birth of the welding material in the fill 2

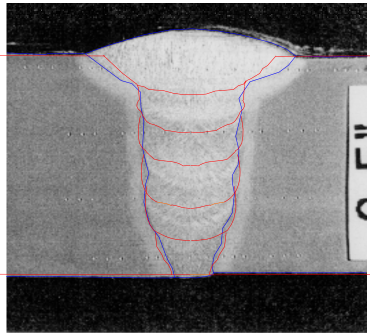
Welding: Temperature evolution during the welding cycle

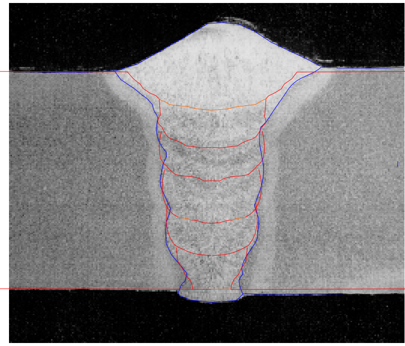


The temperature fields are only presented for times close to the heat input.

A red picture indicates a jump in time.

Liquid pool zone

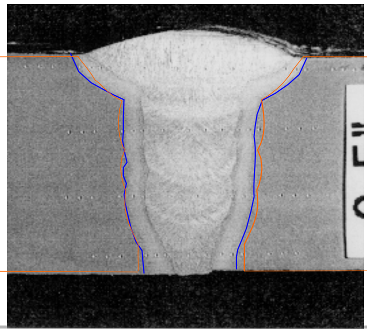


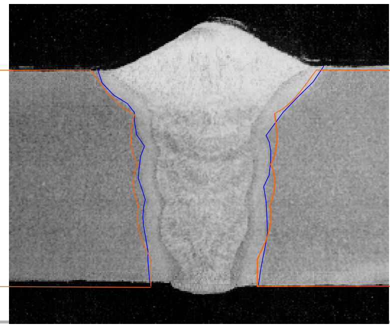


Red numerical values

Blue experimental values

Haz zone





T > 850 °C

Phase transformation from phase g to phase (a + g)

Examples of nonlinear heat transfer problems

Example 1: Calculate the temperature profile at 10s 100s and 1e4s for the infinite cylinder made with two layers of Alumina and Cupper. Consider the variation of the Cupper and the Alumina Conductivity with temperature. Wich are the nonlinear sources?