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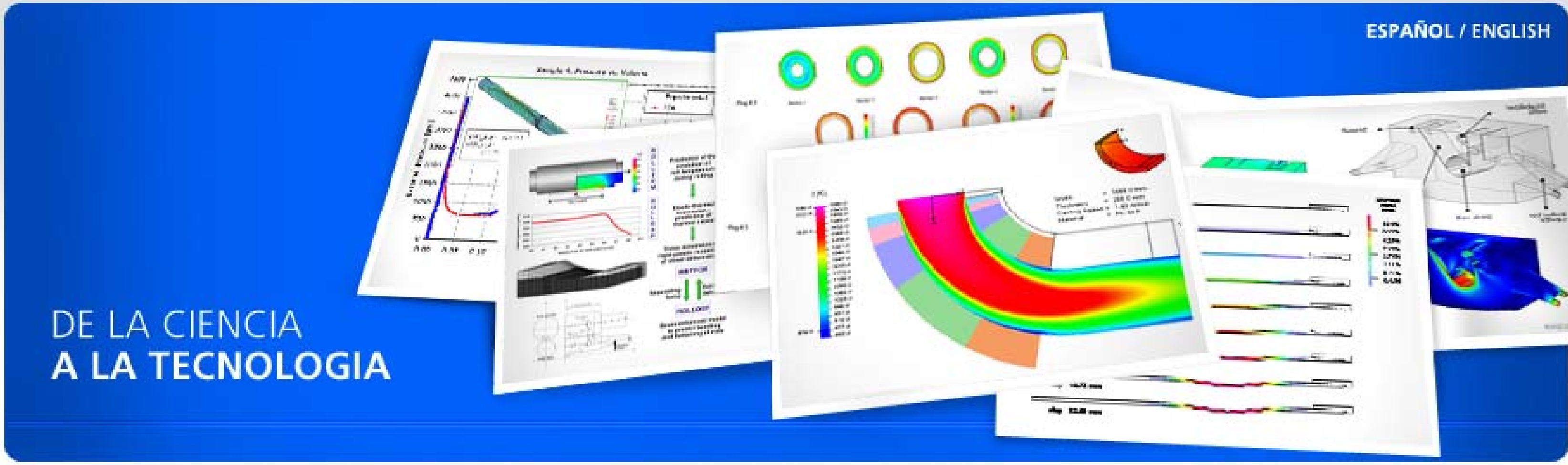

FINITE ELEMENT METHOD IN FLUID DYNAMICS

Part 1: Laminar flow

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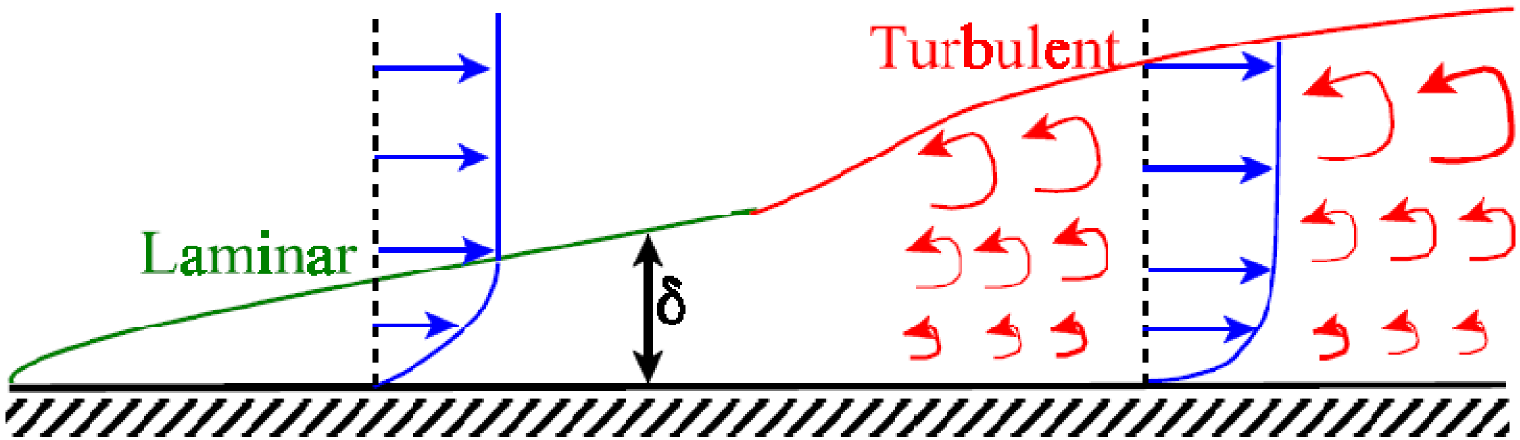
Laminar Flow



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Laminar Flow

Ideal Fluids

- ✓ Fluid with no friction
- ✓ Also referred to as an *inviscid* (zero viscosity) fluid
- ✓ Internal forces at any section within are normal (pressure forces)
- ✓ Practical applications: many flows approximate frictionless flow away from solid boundaries.
- ✓ Do not confuse ideal fluid with a perfect (ideal) gas.

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Real Fluids

- ✓ Tangential or shearing forces always develop where there is motion relative to solid body
- ✓ Shear forces oppose motion of one particle past another
- ✓ Friction forces gives rise to a fluid property called viscosity

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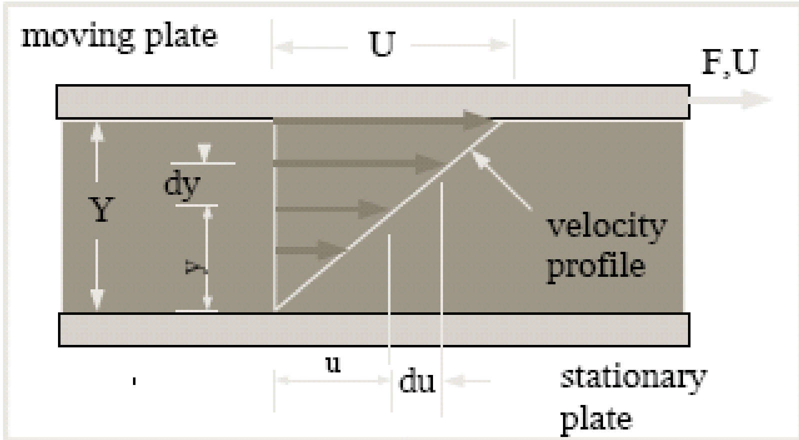
Viscosity



A measure of a fluid's resistance to angular deformation, e.g.,

– Motor oil: high viscosity, feels sticky

– Gasoline: low viscosity, flows "faster"

Friction forces result from cohesion and momentum interchange between molecules.

The shear stress between layers

$$\tau = \mu \frac{du}{dx}$$

μ = *coefficient of viscosity, absolute viscosity,dynamic viscosity,* or simply *viscosity*

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Viscosity

- In B.G. Units

$$[\mu] = \frac{[\tau]}{[du/dy]} = \frac{lb/ft^2}{fps/ft} = \frac{lb\,sec}{ft^2}$$

- In S.I. Units

$$[\mu] = \frac{[\tau]}{[du/dy]} = \frac{N/m^2}{s^{-1}} = \frac{N\,s}{m^2}$$

- The *poise* (P):
  - Metric unit of viscosity
  - Named after Jean Louis Poiseuille (1799-1869)
  - The poise: $1\ P = 0.10\ N{\cdot}s/m^2$
  - The *centipoise*: $1\ cP = 0.01\ P = 1\ mN{\cdot}s/m^2$
  - For water at $68.4°F$ ($20.22°C$), $\mu = 1\ cP$

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Viscosity

- Ratio of absolute viscosity to density

$$\nu = \frac{\mu}{\rho}$$

- Appears in many problems in fluids
- Called *kinematic viscosity* because it involves no force (dynamic) dimensions
- B.G. Units = ft^2/sec, S.I. Units = m^2/s
- The *stoke* (St)
  - Metric unit of kinematic viscosity
  - Named after Sir George Stokes (1819-1903)
  - The *centistoke*: $1\ cSt = 0.01\ St = 10^{-6}\ m^2/s$

μ for most fluids is virtually independent of pressure for the range of interest to engineers
ν for gases varies strongly with pressure because of changes in density (ρ)

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Viscosity

Liquid	Viscosity in mPa.s
Water at 0°C	1.79
Water at 20°C	1.002
Water at 100°C	0.28
Glycerin at 0°C	12070
Glycerin at 20°C	1410
Glycerin at 30°C	612
Glycerin at 100°C	14.8
Mercury at 20°C	1.55
Mercury at 100°C	1.27
Motor Oil SAE 30	200
Motor Oil SAE 60	1000
Ketchup	50,000

Gas	Viscosity in 10^{-6} Pa.s
Air at 100K	7.1
Air at 300K	18.6
Air at 400K	23.1
Hydrogen at 300K	9.0
Helium at 300K	20.0
Oxygen at 300K	20.8
Nitrogen at 300K	17.9
Xenon at 300K	23.2

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Density

The **density** of a material is defined as its mass per unit volume

$$\begin{aligned}\rho(x,y,z) &= \lim_{Volume\to 0}\frac{\text{mass of box}}{\text{volume of box}}\\ &= \lim_{\Delta x,\Delta y,\Delta z\to 0}\left(\frac{m(x+\Delta x,y+\Delta y,z+\Delta z)-m(x,y,z)}{\Delta x\Delta y\Delta z}\right)\\ &= \frac{dm}{dV}.\end{aligned}$$

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Density

Material	ρ in kg/m^3	Notes
Water	1000	At STP
Iron	7874	Near room temperature
Copper	8920 – 8960	Near room temperature
Lead	11340	Near room temperature
Gold	19300	Near room temperature
Platinum	21450	Near room temperature
Air	1.184	Near room temperature

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Mass conservation equation

"Mass cannot disappear or created from the system"

$$\frac{D\rho}{Dt} + \rho \frac{\partial v_i}{\partial x_i} = 0 \qquad \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0$$

Incompressible flow

$$\frac{\partial v_i}{\partial x_i} = 0 \Rightarrow \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \Rightarrow \nabla \cdot \underline{v} = 0$$

- The incompressibility condition does not imply density uniform (∂ρ/∂xi=0) or constant (∂ρ/∂t=0). Flow of two fluids of different density.
- A flow can be consider incompressible when the fluid particle velocities are lesser than the sonid velocity (Mach number << 1).

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Mass conservation equation

Cartesian coordinates

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho\, v_x)+\frac{\partial}{\partial y}(\rho\, v_y)+\frac{\partial}{\partial z}(\rho\, v_z)=0$$

Cylindrical coordinates

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)+\frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta}+\frac{\partial(\rho v_z)}{\partial z}=0$$

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Momentum Conservation

$$\rho \frac{D v_i}{D t} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i$$

Newton´s Second Law: “ The momentum principle for a collection of particles states that the time rate of change of the total momentum of a given set of particles equals the vector sum of all the external forces acting on the particles of the set provided Newton´s Third Law of action and reaction governs the internal forces. The continuum form of this principle is a basic postulate of continuum mechanics”

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Momentum Conservation

The ***Newtonian model*** of fluid response is based on three assumptions:

(a) shear stress is proportional to the *rate* of shear strain in a fluid particle;

(b) shear stress is zero when the rate of shear strain is zero;

(c) the stress to rate-of-strain relation is isotropic—that is, there is no preferred orientation in the fluid.

A Newtonian fluid is the simplest type of viscous fluid, just like an elastic solid (where stresses are proportional to strains) is the simplest type of deformable solid.

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Momentum Conservation

$$\sigma_{ij} = -\left(p + \frac{2}{3}\mu \,\underline{\nabla} \cdot \underline{v} \right) \delta_{ij} + 2\mu S_{ij}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

For incompressible flow

$$\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v} = -\nabla p + \nabla \cdot \left[\mu \left(\nabla \underline{v} + \nabla \underline{v}^T \right) \right] + \rho \underline{f}$$

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Momentum Conservation

For incompressible flow and constant viscosity

$$\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v} = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{f}$$

Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho f_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho f_y$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho f_z$$

Cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) =$$

$$-\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho f_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) =$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho f_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) =$$

$$-\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho f_z$$

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Momentum Conservation

The pressure does not have an evolution equation. In each time instant, the pressure is adjusted to fulfill the incompressibility condition.

The viscosity is a fluid property, depend only to the temperature for Newtonian fluids.

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Non-dimensional numbers

$$Re = \frac{\rho V L}{\mu} \qquad \frac{inertial \quad forces}{viscous \quad forces}$$

$$Fr = \frac{V^2}{g L} \qquad \frac{inertial \quad forces}{gravitatio \ nal \quad forces}$$

$$Ma = \frac{V}{c} \qquad \frac{fluid \ \ speed}{speed \ \ of \ \ the \ sound \ \ in \ the \ \ fluid}$$

Laminar Flow

Non-dimensional numbers

$$Eu = \frac{\Delta P}{\frac{1}{2}\rho V^2} \qquad \frac{pressure \quad forces}{inertial \quad forces}$$

$$We = \frac{V}{\sqrt{\frac{\sigma}{\rho L}}} \qquad \frac{inertial \quad forces}{surface \quad tension}$$

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Boundary Conditions

In the absence of surface tension, the boundary conditions consistent with the continuum hypothesis are that

(a) the velocity components
(b) the stress tensor components must be everywhere continuous,

including across phase interfaces like the boundaries between the fluid and a solid and between two immiscible fluids.

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Boundary Conditions

- Esential BC: velocities imposed $v_i = w_{imp}$
- Natural BC: traction imposed $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t} \Rightarrow \mu \nabla \underline{v} \cdot \underline{n} - p\underline{n} = \underline{t}$

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Energy Conservation

$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p \underline{v} \cdot \underline{\nabla} T = \underline{\nabla} \cdot \left(\underline{\underline{k}} \cdot \underline{\nabla} T \right) + q_V$$

Transient term — Convective term — Diffusive term — Volumetric heat [W/m³]

ρ density [Kg/m³]

C_p Specific heat [J/Kg °K]

$\underline{\underline{k}}$ Thermal conductivity [W/M°K]

$\underline{v}$ Velocity [m/s]

$$\underline{q} = -k \, \underline{\nabla} T$$

$\underline{q}$: heat flux [W/m2]

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Specific heat capacity

Substance	Phase	C_p J/(g·K)	$C_{p,m}$ J/(mol·K)	$C_{v,m}$ J mol^{-1}·K^{-1}	Volumetric heat capacity J/(cm^3·K)
Air (Sea level, dry, 0 °C)	gas	1.0035	29.07	20.7643	0.001297
Air (typical room conditions^A)	gas	1.012	29.19	20.85	
Aluminium	solid	0.897	24.2		2.422
Carbon dioxide CO_2	gas	0.839*	36.94	28.46	
Copper	solid	0.385	24.47		3.45
Ethanol	liquid	2.44	112		1.925
Gasoline	liquid	2.22	228		1.64
Glass	solid	0.84			
Gold	solid	0.129	25.42		2.492
Iron	solid	0.450	25.1		3.537
Lead	solid	0.127	26.4		1.44
Mercury	liquid	0.1395	27.98		1.888
Methane at 2 °C	gas	2.191			
Water at 100 °C (steam)	gas	**2.080**	**37.47**	**28.03**	
Water at 25 °C	liquid	**4.1813**	**75.327**	**74.53**	**4.186**
Water at −10 °C (ice)	solid	**2.05**	**38.09**		**1.938**

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Thermal conductivity

Material	Thermal conductivity [W/(m·K)]
Air	0.025
Water (liquid)	0.6
Glass	1.1
Soil	1.5
Concrete stone	1.7
Ice	2
Sandstone	2.4
Stainless steel	12.11 ~ 45.0
Lead	35.3
Aluminium	237 (pure) 120—180 (alloys)
Gold	318
Copper	401
Silver	429
Diamond	900 - 2320
Graphene	(4840±440) - (5300±480)

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Energy Conservation – Non dimensional numbers

$$Gr = \frac{g\ \beta\left(T_s - T_\infty\right)L^3}{V^2} \qquad \frac{bouyancy\quad forces}{viscous\quad forces}$$

β = volumetric thermal expansion coefficient (equal to approximately 1/T, for ideal fluids, where T is absolute temperature)

$$\Pr = \frac{Cp\ \mu}{k} \qquad \frac{viscous\quad diffusion\quad rate}{thermal\quad diffusion\quad rate}$$

$$Br = \frac{\mu\, V^2}{k\left(T_{wall} - T_{bulk}\right)} \qquad \frac{heat\ \ production\ \ by\ viscous\ \ dissipatio\ n}{heat\ \ production\ \ by\ conduction}$$

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Energy Conservation – Non dimensional numbers

$$Ec = \frac{Br}{\Pr} = \frac{V^2}{Cp\ \Delta T} \qquad \frac{Kinetic\ \ energy}{Enthalpy}$$

$$Nu = \frac{h\ L}{k_f} \qquad \frac{convective\ \ heat\ transfer}{conductive\ \ heat\ \ transfer}$$