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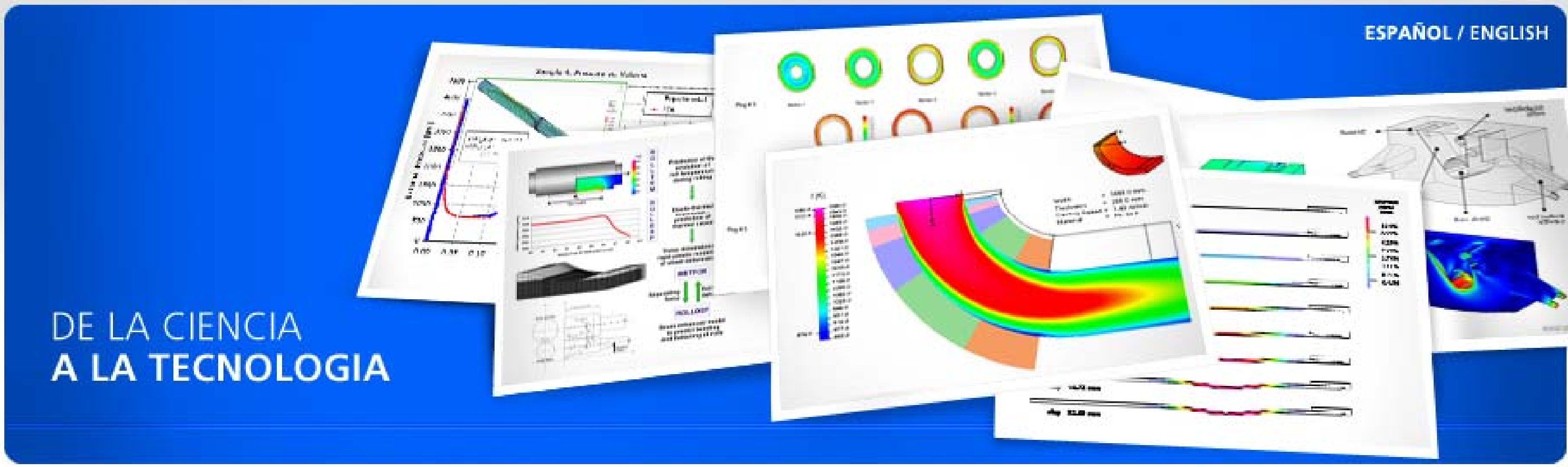
Simulación y Tecnología

Simulation and Technology



FINITE ELEMENT METHOD IN FLUID DYNAMICS

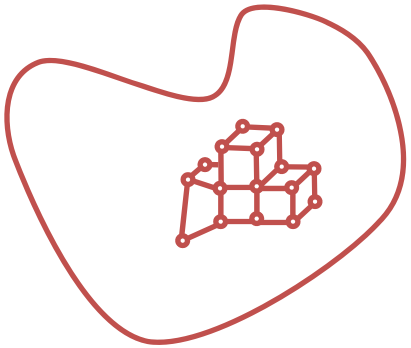
Part 2: The finite element method

Marcela B. Goldschmit

The finite element method

2D / 3D Problems

Elements and nodes



The interpolation functions inside an element

$$\tilde{V} = \sum_{1}^{NNOD} h_k \, V_k$$

V_k : variable at node "k", examples v_x, v_y, v_z, p, T

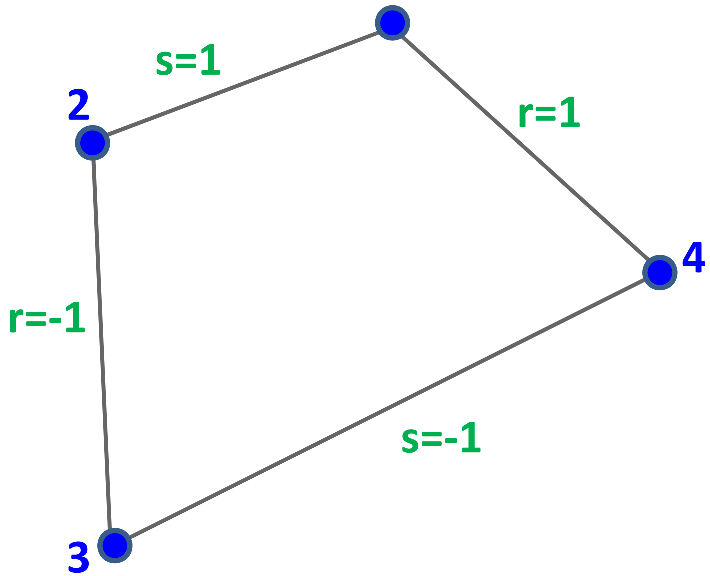
h_k : interpolation function

$h_k = 1$ at node "k"

$h_k = 0$ at node $\neq$ "k"

The finite element method

2D example



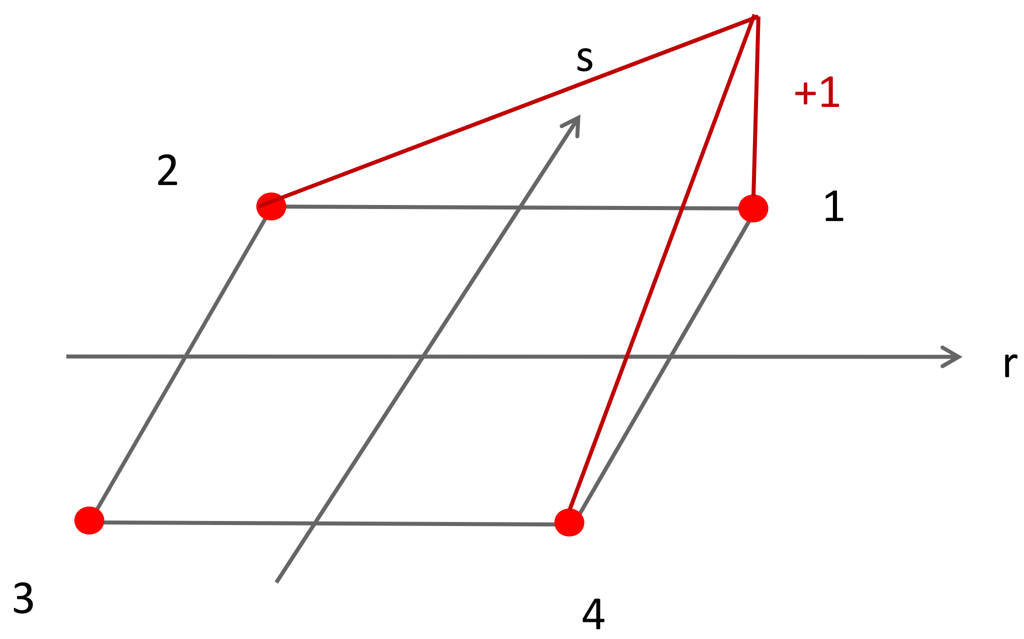
Natural coordinate system inside each element (r, s)

$$
\begin{aligned}
-1 &\leq r \leq 1 \\
-1 &\leq s \leq 1
\end{aligned}
$$

$$
\begin{aligned}
h_1(r,s) &= \frac{1}{4}(1+r)(1+s) \\
h_2(r,s) &= \frac{1}{4}(1-r)(1+s) \\
h_3(r,s) &= \frac{1}{4}(1-r)(1-s) \\
h_4(r,s) &= \frac{1}{4}(1+r)(1-s)
\end{aligned}
$$

The finite element method

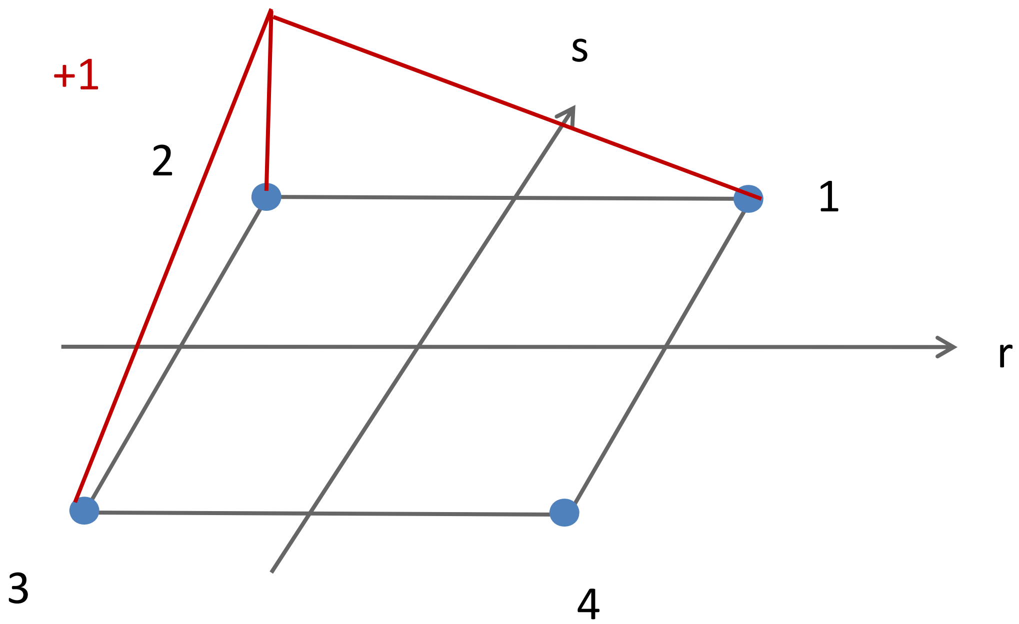
2D four – nodes element



h1

The finite element method

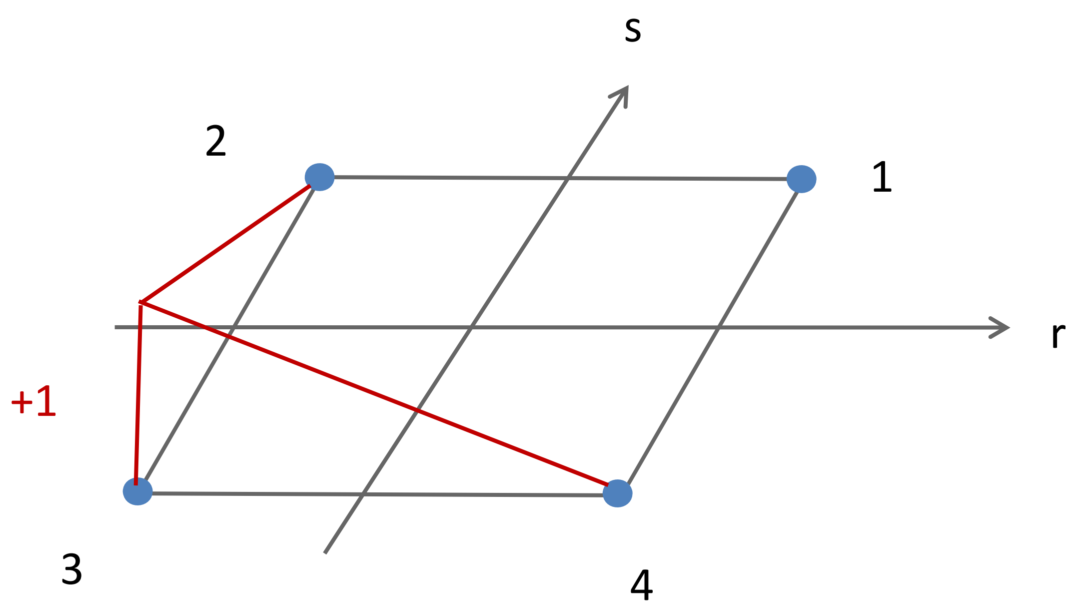
2D four – nodes element



h2

The finite element method

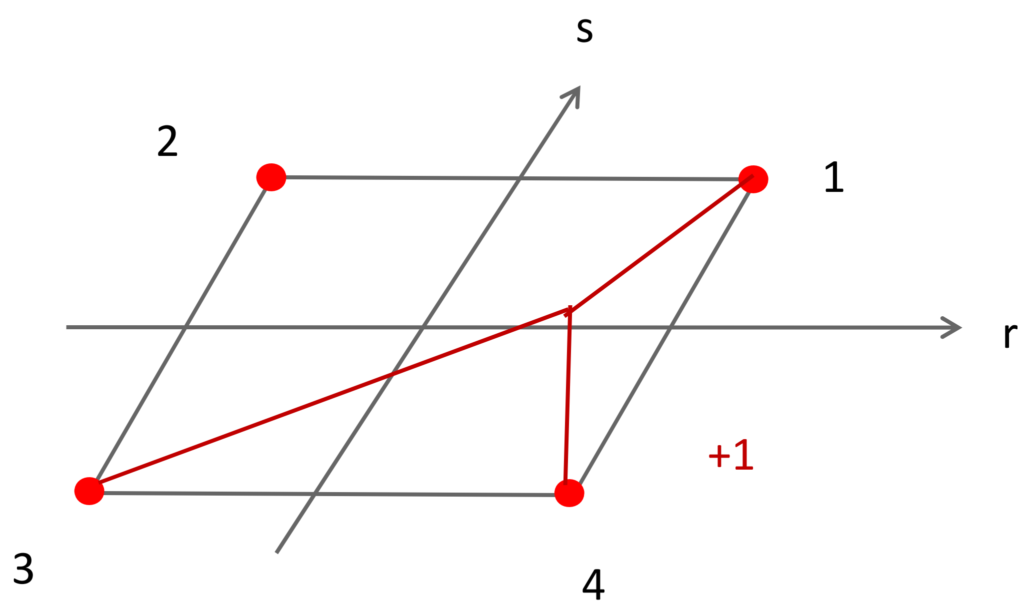
2D four – nodes element



h3

The finite element method

2D four – nodes element



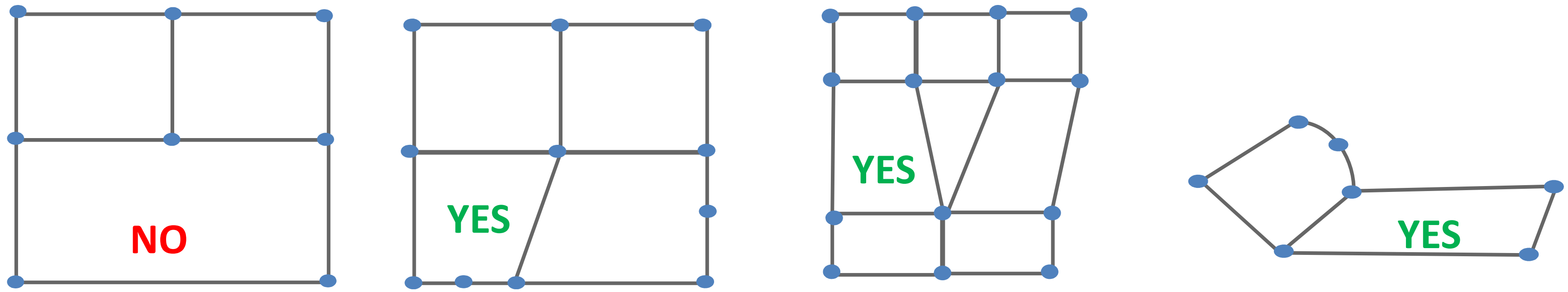
h4

The finite element method

$$\sum_{k=1}^{k=NNOD} h_k = 1$$

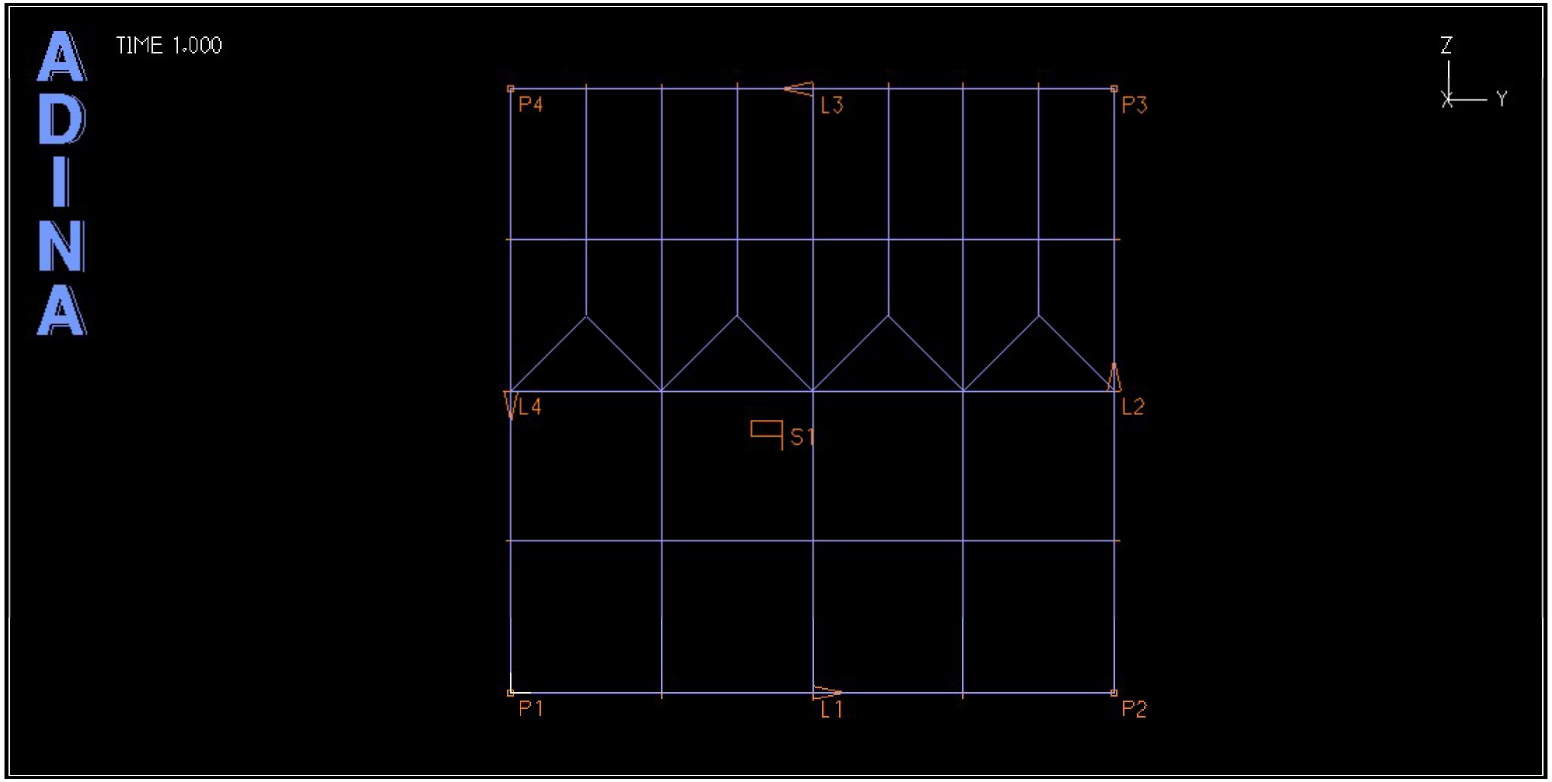
to be able to represent constant temperature situations

Are these meshes acceptable?



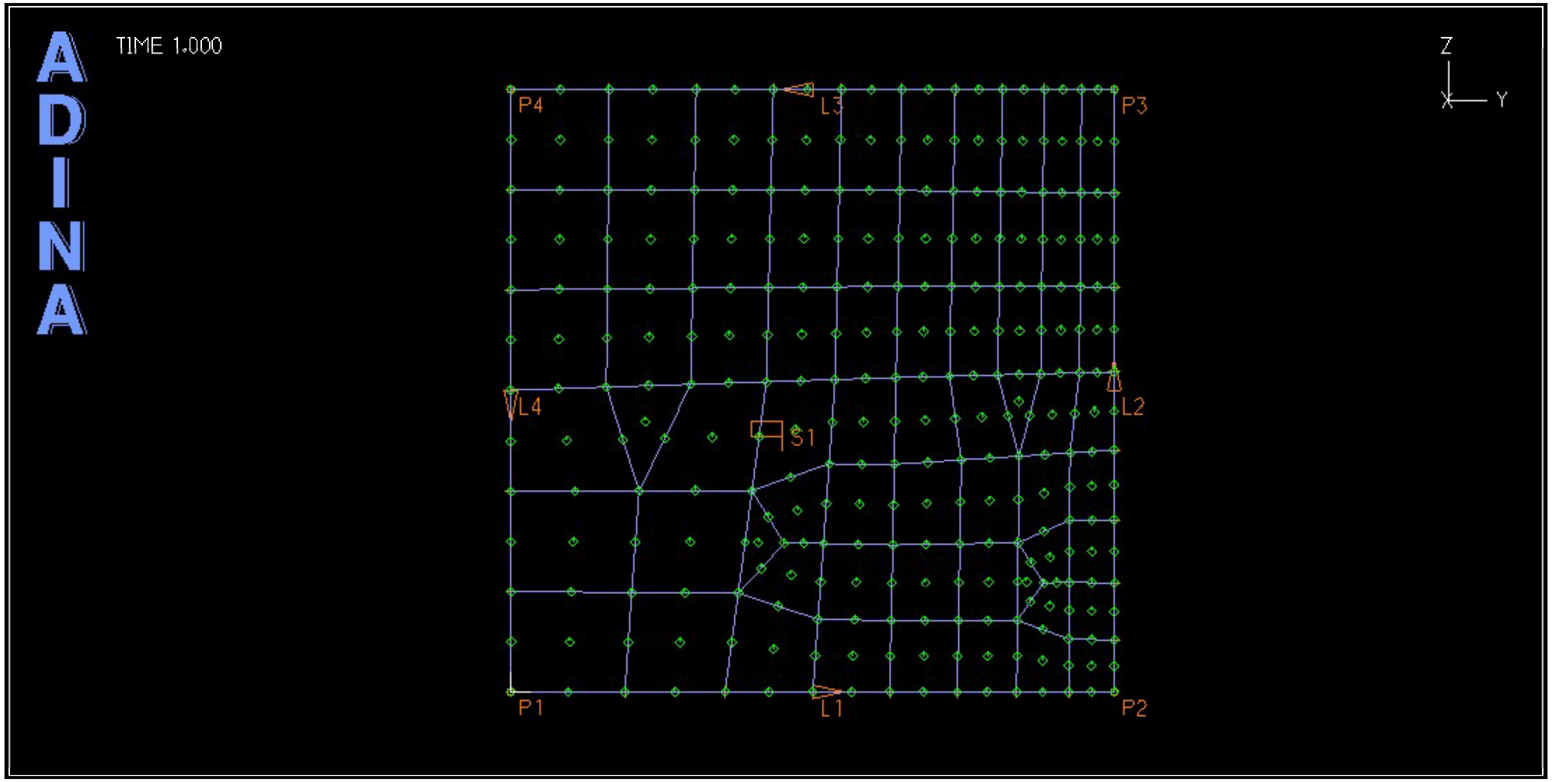
The finite element method

Good Mesh



The finite element method

Good Mesh



The finite element method

Good Mesh

However!!!

Minimize the element distortions to have good predictive capability

Target for each element: det($\underline{J}$)=const

The finite element method

Isoparametric elements

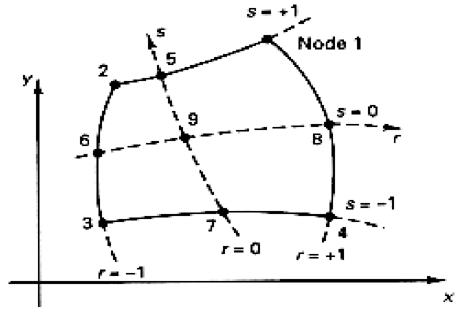
$$\tilde{T}(r,s,t) = h_k(r,s,t)\,T_k$$

$$x(r,s,t) = h_k(r,s,t)\,x_k$$

$$y(r,s,t) = h_k(r,s,t)\,y_k$$

$$z(r,s,t) = h_k(r,s,t)\,z_k$$

The finite element method



(a) 4 to 9 variable-number-nodes two-dimensional element

Include only if node i is defined

		$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$					$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1-s^2)(1-r)$					$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$					$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1-s^2)(1+r)$					$-\frac{1}{2}h_9$
$h_9 =$	$(1-r^2)(1-s^2)$					

(b) Interpolation functions

Figure 5.4 Interpolation functions of four to nine variable-number-nodes two-dimensional element

From Bathe, Finite Element Procedures

The finite element method

$$h_1 = g_1 - (g_9 + g_{12} + g_{17})/2 \qquad h_6 = g_6 - (g_{13} + g_{14} + g_{18})/2$$
$$h_2 = g_2 - (g_9 + g_{10} + g_{18})/2 \qquad h_7 = g_7 - (g_{14} + g_{15} + g_{19})/2$$
$$h_3 = g_3 - (g_{10} + g_{11} + g_{19})/2 \qquad h_8 = g_8 - (g_{15} + g_{16} + g_{20})/2$$
$$h_4 = g_4 - (g_{11} + g_{12} + g_{20})/2 \qquad h_j = g_j \text{ for } j = 9, \ldots, 20$$
$$h_5 = g_5 - (g_{13} + g_{16} + g_{17})/2$$

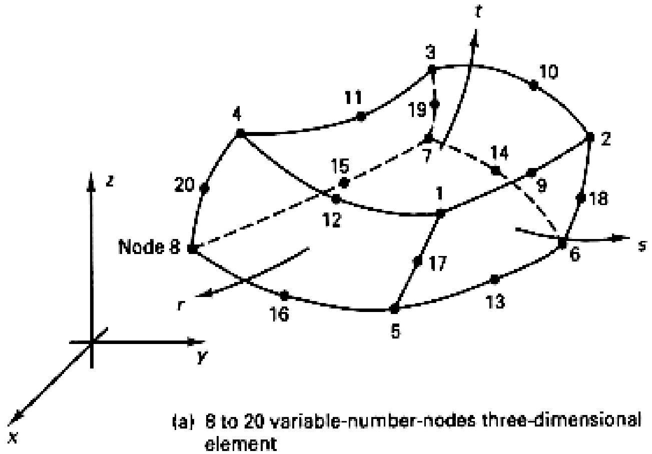
$g_i = 0$ if node i is not included; otherwise,

$$g_i = G(r, r_i)\, G(s, s_i)\, G(t, t_i)$$

$$\left.\begin{aligned} G(\beta, \beta_i) &= \tfrac{1}{2}(1 + \beta_i \beta) && \text{for } \beta_i = \pm 1 \\ G(\beta, \beta_i) &= (1 - \beta^2) && \text{for } \beta_i = 0 \end{aligned}\right\} ;\ \beta = r, s, t$$

(b) Interpolation functions

Figure 5.5 *(continued)*



(a) 8 to 20 variable-number-nodes three-dimensional element

Figure 5.5 Interpolation functions of eight to twenty variable-number-nodes three-dimensional element

From Bathe, Finite Element Procedures

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FEM heat transfer

$$\underline{\underline{\mathbf{M}}} \cdot \dot{\underline{\widehat{\mathbf{T}}}} + \left(\underline{\underline{\mathbf{N}}} + \underline{\underline{\mathbf{K}}} \right) \cdot \underline{\widehat{\mathbf{T}}} = \underline{\mathbf{F}}$$

$$\underline{\underline{\mathbf{M}}} = \sum_{e=1}^{NE} \underline{\underline{\mathbf{M}}}^{G(e)} \quad ; \quad \underline{\underline{\mathbf{N}}} = \sum_{e=1}^{NE} \underline{\underline{\mathbf{N}}}^{G(e)} \quad ;$$

$$\underline{\underline{\mathbf{K}}} = \sum_{e=1}^{NE} \underline{\underline{\mathbf{K}}}^{G(e)} \quad ; \quad \underline{\mathbf{F}} = \sum_{e=1}^{NE} \underline{\mathbf{F}}^{G(e)}$$

$$M_{ij}^{G(e)} = \int_{\Omega^e} h_i \, \rho C_p h_j \, d\Omega$$

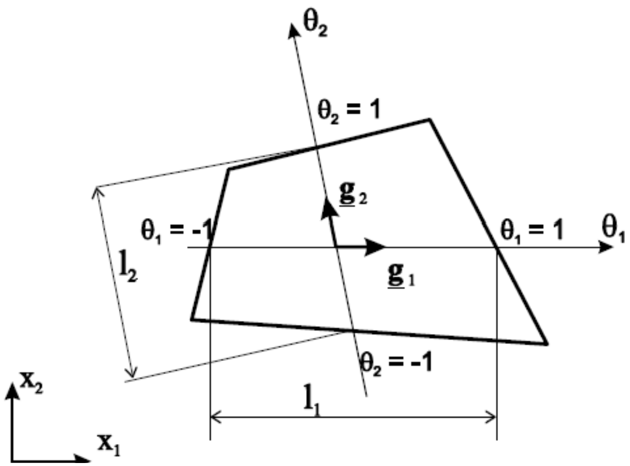
$$N_{ij}^{G(e)} = \int_{\Omega^e} h_i \, \rho C_p \underline{\mathbf{v}} \cdot \underline{\boldsymbol{\nabla}} h_j \, d\Omega$$

$$K_{ij}^{G(e)} = \int_{\Omega^e} \underline{\boldsymbol{\nabla}} h_i \cdot \underline{\underline{\mathbf{k}}} \cdot \underline{\boldsymbol{\nabla}} h_j \, d\Omega$$

$$F_i^{G(e)} = \int_{\Omega^e} h_i \, q_v \, d\Omega + \int_{\Gamma_q^{(e)}} h_i \, q_n^* \, d\Gamma$$

The Galerkin method is not good for $Pe \succeq 1$; $Pe^{(e)} = \frac{v \; L^{(e)}}{2 \; k}$

FEM heat transfer



SUPG: Streamline Upwind Petrov Galerkin Method : The weighted functions are different of the approximation functions

$$W_i = \tau\, \underline{\mathbf{v}} \cdot \underline{\nabla} h_i \quad ; \quad \tau = \sum_{i=1}^{ND} \frac{\alpha_i\, v_i^c\, l_i}{2} \frac{1}{\|\underline{\mathbf{v}}\|^2} \quad ;$$

$$\alpha_i = \coth |Pe_i| - \frac{1}{|Pe_i|} \quad ; \quad Pe_i = \frac{v_i\, l_i}{2\, k}$$

$v_i^c = \underline{\mathbf{g}}_i \cdot \underline{\mathbf{v}}^c$; $\underline{\mathbf{v}}^c =$ is the central velocity ;

$ND =$ number of dimension ; $l_i =$ characteristics dimension

$$\|\underline{\mathbf{v}}\| = \left[\sum_{i=1}^{ND} v_i^2 \right]^{1/2} \qquad \underline{\mathbf{H}}^T = (h_1, h_2, ..., h_{nnodo})$$

FEM heat transfer

$$\underline{\underline{\mathbf{M}}} \cdot \dot{\widehat{\underline{\mathbf{T}}}} + \left(\underline{\underline{\mathbf{N}}} + \underline{\underline{\mathbf{K}}} \right) \cdot \widehat{\underline{\mathbf{T}}} = \underline{\mathbf{F}}$$

$$\underline{\underline{\mathbf{M}}} = \sum_{e=1}^{NE} \left(\underline{\underline{\mathbf{M}}}^{G^{(e)}} + \underline{\underline{\mathbf{M}}}^{P^{(e)}} \right) \quad ; \quad \underline{\underline{\mathbf{N}}} = \sum_{e=1}^{NE} \left(\underline{\underline{\mathbf{N}}}^{G^{(e)}} + \underline{\underline{\mathbf{N}}}^{P^{(e)}} \right)$$

$$\underline{\underline{\mathbf{K}}} = \sum_{e=1}^{NE} \left(\underline{\underline{\mathbf{K}}}^{G^{(e)}} + \underline{\underline{\mathbf{K}}}^{P^{(e)}} \right) \quad ; \quad \underline{\mathbf{F}} = \sum_{e=1}^{NE} \left(\underline{\mathbf{F}}^{G^{(e)}} + \underline{\mathbf{F}}^{P^{(e)}} \right)$$

$$M_{ij}^{P^{(e)}} = \int_{\Omega^e} W_i \, \rho C_p h_j \, d\Omega \qquad N_{ij}^{P^{(e)}} = \int_{\Omega^e} W_i \, \rho C_p \underline{\mathbf{v}} \cdot \underline{\boldsymbol{\nabla}} h_j \, d\Omega$$

$$K_{ij}^{P^{(e)}} = - \int_{\Omega^e} W_i \, \underline{\boldsymbol{\nabla}} \cdot \left(\underline{\underline{\mathbf{k}}} \cdot \underline{\boldsymbol{\nabla}} h_j \right) \, d\Omega \qquad F_i^{P^{(e)}} = \int_{\Omega^e} W_i \, q_v \, d\Omega$$

Laminar Flow

$$
\begin{aligned}
\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \underline{\nabla v} &= \underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{b} \\
\underline{\nabla} \cdot \underline{v} &= 0 \\
\underline{\underline{\sigma}} &= -p\underline{\underline{I}} + \mu \left(\underline{\nabla v} + \underline{\nabla v^T} \right)
\end{aligned}
$$

Boundary conditions

$$
\begin{aligned}
\underline{v} &= \underline{w} \quad \text{in } \Gamma^D \\
\underline{\underline{\sigma}} \cdot \underline{n} &= \underline{t} \quad \text{in } \Gamma^N
\end{aligned}
$$

$$\Gamma^N \cup \Gamma^D = \partial\Omega \text{ y } \Gamma^N \cap \Gamma^D = \oslash$$

Initial conditions

$$\underline{v} = \underline{v}_0 \quad \text{in } \Omega,\ t = 0$$

$$\underline{\nabla} \cdot \underline{v}_0 = 0 \text{ y } \underline{v}_0 = \underline{w}_0 \text{ in } \Gamma^D$$

Laminar Flow

Non-linearity

Picard Method

$$v_i \frac{\partial v_j}{\partial x^i} \approx v_i^k \frac{\partial v_j^{k-1}}{\partial x^i}$$

Newton-Raphson Method

$$v_i \frac{\partial v_j}{\partial x^i} \approx v_i^{k-1} \frac{\partial v_j^k}{\partial x^i} + v_i^k \frac{\partial v_j^{k-1}}{\partial x^i} - v_i^{k-1} \frac{\partial v_j^{k-1}}{\partial x^i}$$

Laminar Flow

Galerkin Method – Newton Raphson Method

$$
\begin{aligned}
v_i &= h^J_{v_i}\widehat{V}^J \qquad & u_i = h^I_{v_i}\widehat{U}^I \\
p &= h^I_p\widehat{P}^I \qquad & q = h^I_p\widehat{Q}^I
\end{aligned}
$$

$$
\left[\int_\Omega \rho h^I_{v_i} h^J_{v_i} d\Omega\right]\frac{d\widehat{V}^J}{dt}+
$$

$$
+\left[\int_\Omega \rho\left(\frac{\partial v_i}{\partial x^j}\right)^{k-1} h^I_{v_i}\ h^J_{v_j} d\Omega\right]\widehat{V}^J + \left[\int_\Omega \rho v_j^{k-1} h^I_{v_i}\ \frac{\partial h^J_{v_i}}{\partial x^j} d\Omega\right]\widehat{V}^J
$$

$$
+\left[\int_\Omega \mu \frac{\partial h^I_{v_i}}{\partial x^j}\left(\frac{\partial h^J_{v_i}}{\partial x^j}+\frac{\partial h^J_{v_j}}{\partial x^i}\right) d\Omega\right]\widehat{V}^J - \left[\int_\Omega \frac{\partial h^I_{v_i}}{\partial x^i} h^J_p d\Omega\right]\widehat{P}^J \quad =
$$

$$
+\left[\int_\Omega h^I_{v_i} b_i d\Omega\right] + \left[\int_{\Gamma^N} h^I_{v_i} t_i d\Gamma^N\right] + \left[\int_\Omega \rho h^I_{v_i}\left(v_j\frac{\partial v_i}{\partial x^j}\right)^{k-1} d\Omega\right]
$$

$$
\left[\int_\Omega h^I_p \frac{\partial h^J_{v_i}}{\partial x^i} d\Omega\right]\widehat{V}^J \quad = \quad 0
$$

Laminar Flow

- SUPG Method $h^I_i + w^I_i$
- VP formulation

$$
\begin{aligned}
\underline{\underline{M}} \cdot \frac{d\underline{\widehat{V}}}{dt} + \left(\underline{\underline{N}} + \underline{\underline{K}}\right) \cdot \underline{\widehat{V}} + \underline{\underline{G}} \cdot \underline{\widehat{P}} &= \underline{R} \\
\underline{\underline{G}}^T \cdot \underline{\widehat{V}} &= 0
\end{aligned}
$$

$$
\begin{aligned}
M_{IJ} &= \left[\int_\Omega \rho \left(h^I_{v_i} + w^I_{v_i}\right) h^J_{v_i} d\Omega\right] \\
N_{IJ} &= \left[\int_\Omega \rho \left(\frac{\partial v_i}{\partial x^j}\right)^{k-1} \left(h^I_{v_i} + w^I_{v_i}\right) h^J_{v_j} d\Omega\right] + \left[\int_\Omega \rho\, v_j^{k-1} \left(h^I_{v_i} + w^I_{v_i}\right) \frac{\partial h^J_{v_i}}{\partial x^j} d\Omega\right] \\
K_{IJ} &= \left[\int_\Omega \mu \frac{\partial h^I_{v_i}}{\partial x^j} \left(\frac{\partial h^J_{v_i}}{\partial x^j} + \frac{\partial h^J_{v_j}}{\partial x^k}\right) d\Omega\right] - \left[\int_\Omega \mu w^I_{v_i} \frac{\partial}{\partial x^j} \left(\frac{\partial h^J_{v_i}}{\partial x^j} + \frac{\partial h^J_{v_j}}{\partial x^k}\right) d\Omega\right] \\
G_{IJ} &= -\left[\int_\Omega \frac{\partial h^I_{v_i}}{\partial x^i} h^J_p d\Omega\right] + \left[\int_\Omega w^I_{v_i} \frac{\partial h^J_p}{\partial x^i} d\Omega\right] \\
R_I &= \left[\int_\Omega \left(h^I_{v_i} + w^I_{v_i}\right) b_i d\Omega\right] + \left[\int_\Omega \left(h^I_{v_i} + w^I_{v_i}\right) \left(v_j \frac{\partial v_i}{\partial x^j}\right)^{k-1} d\Omega\right] + \left[\int_{\Gamma^N} h^I_{v_i} t_i d\Gamma^N\right]
\end{aligned}
$$

Laminar Flow

- SUPG Method
- Penalization formulation $\lambda \gg \mu$ $\quad p = -\lambda \underline{\nabla} \cdot \underline{v}$

$$\int_\Omega q p \, d\Omega = -\lambda \int_\Omega q \underline{\nabla} \cdot \underline{v} \, d\Omega$$

$$\underline{\underline{M_p}} \cdot \underline{\widehat{P}} = \lambda \underline{\underline{G}}^T \cdot \underline{\widehat{V}}$$

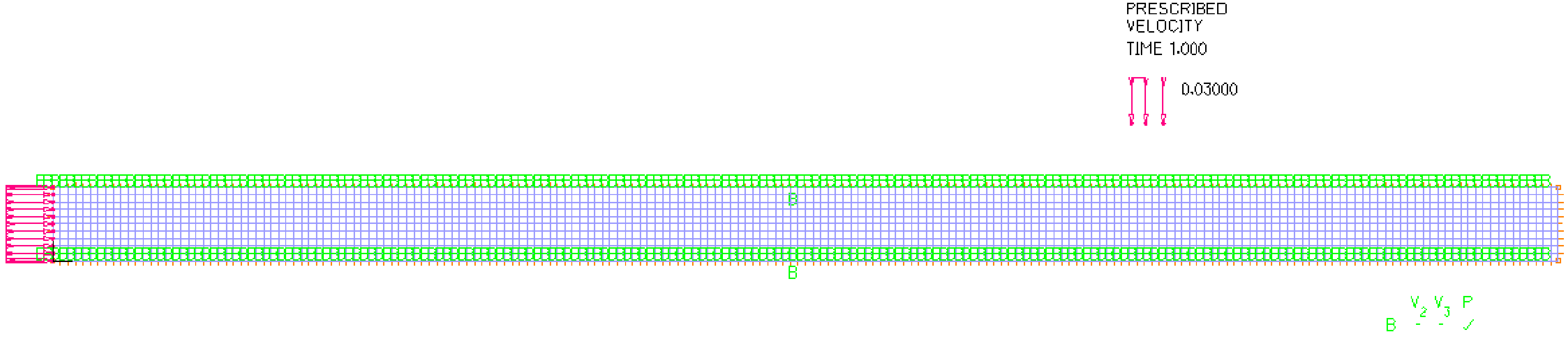
$$M_{p\ JK} = \left[\int_\Omega h_p^J h_p^K \, d\Omega \right]$$

$$\underline{\widehat{P}} = \lambda \underline{\underline{M_p}}^{-1} \cdot \underline{\underline{G}}^T \cdot \underline{\widehat{V}}$$

$$\underline{\underline{M}} \cdot \frac{d\underline{\widehat{V}}}{dt} + \left(\underline{\underline{N}} + \underline{\underline{K}} + \lambda \underline{\underline{G}} \cdot \underline{\underline{M_p}}^{-1} \cdot \underline{\underline{G}}^T \right) \cdot \underline{\widehat{V}} = \underline{R}$$

Channel

Geometry

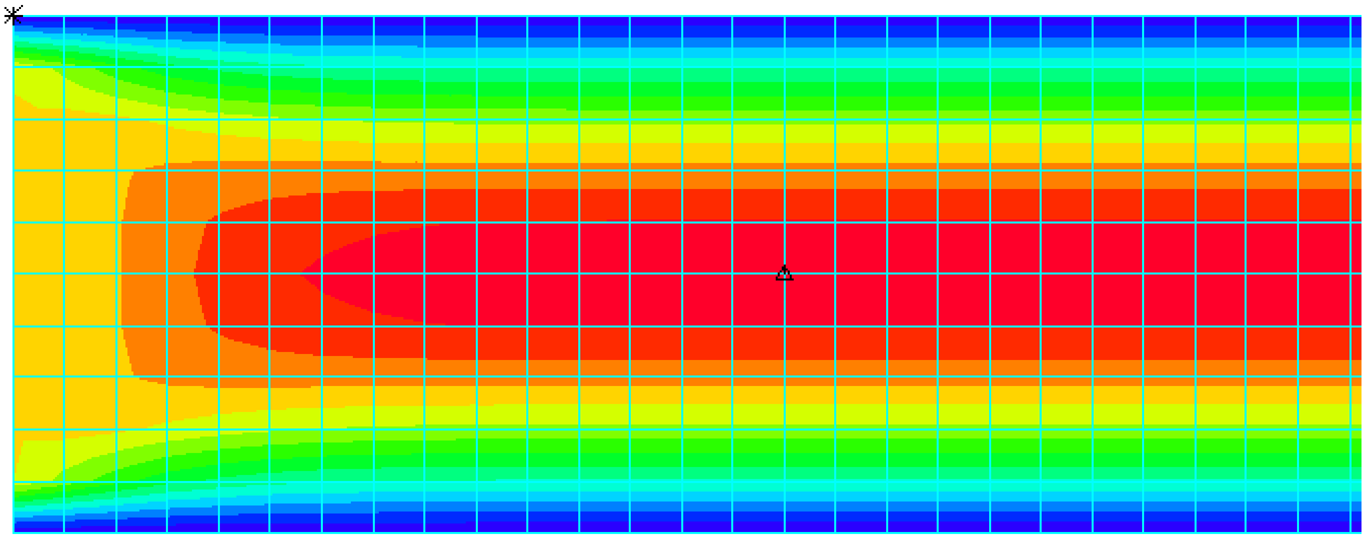


Boundary conditions at the walls: non-slip

Boundary conditions at the entrance: velocities

Channel

Laminar flow

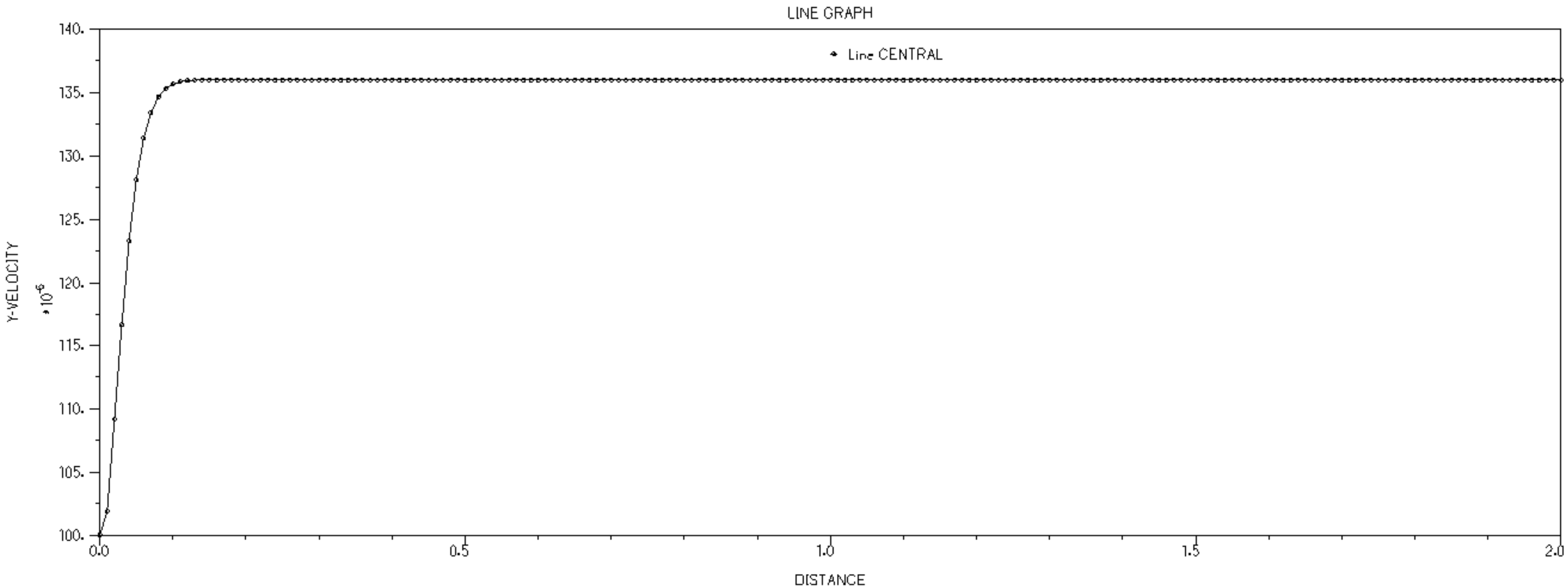


Velocity = 10e-4 m/s
Reynolds = 10

Channel

Laminar flow

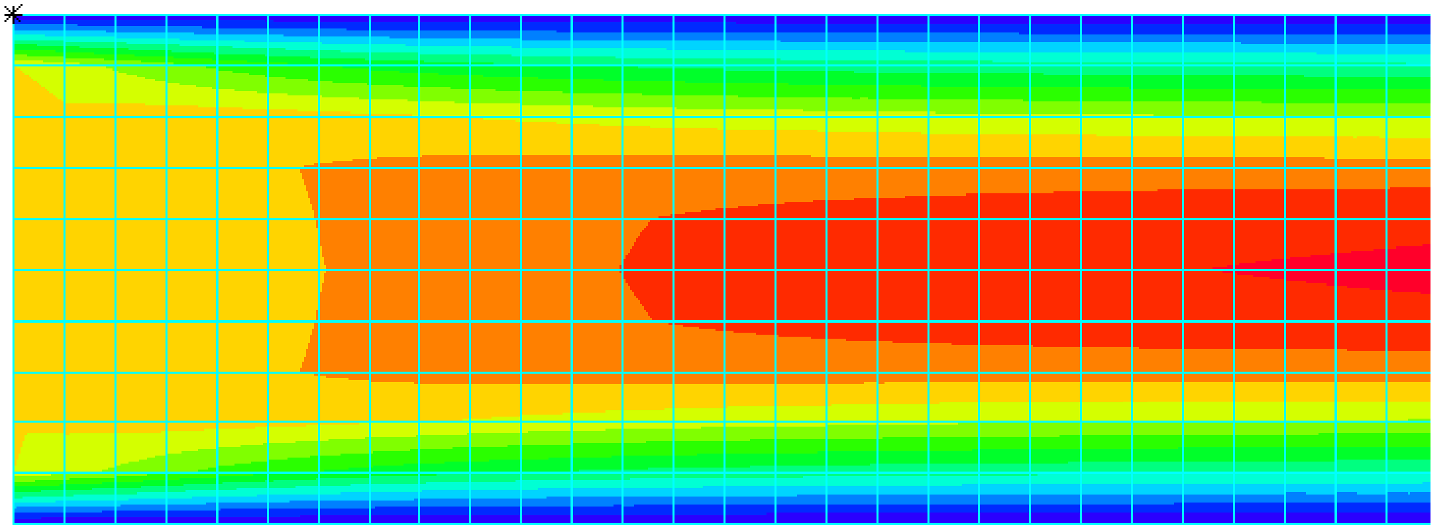
Velocities in the central line



Velocity = 10e-4 m/s
Reynolds = 10

Channel

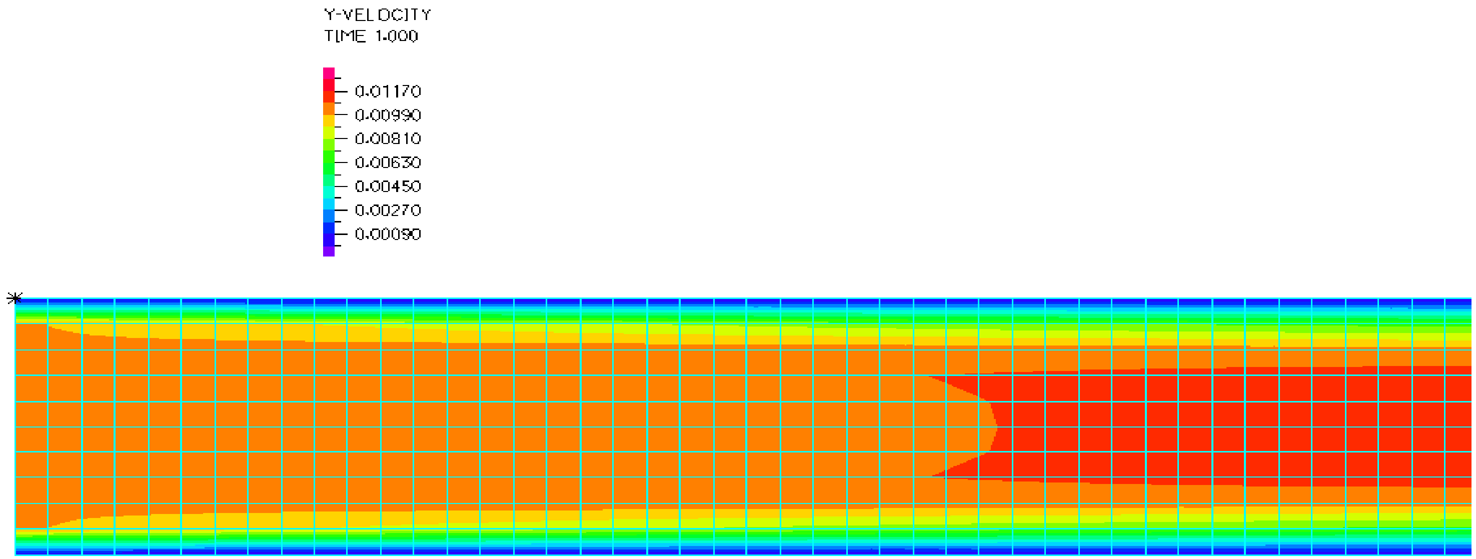
Laminar flow



Velocity = 10e-3 m/s
Reynolds = 100

Channel

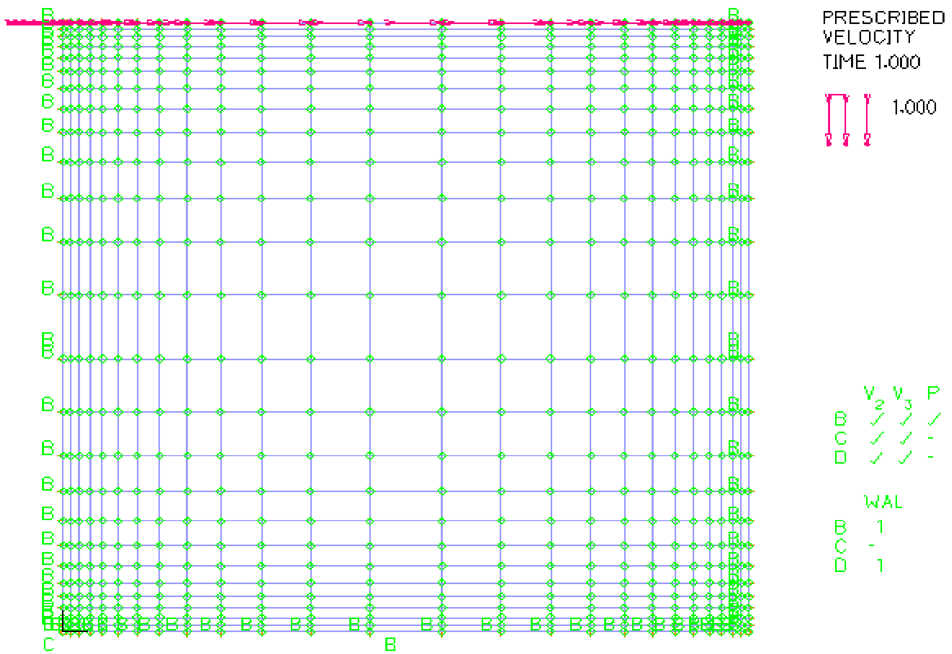
Laminar flow



Velocity = 0.01 m/s
Reynolds = 1000

Cavity with sliding wall

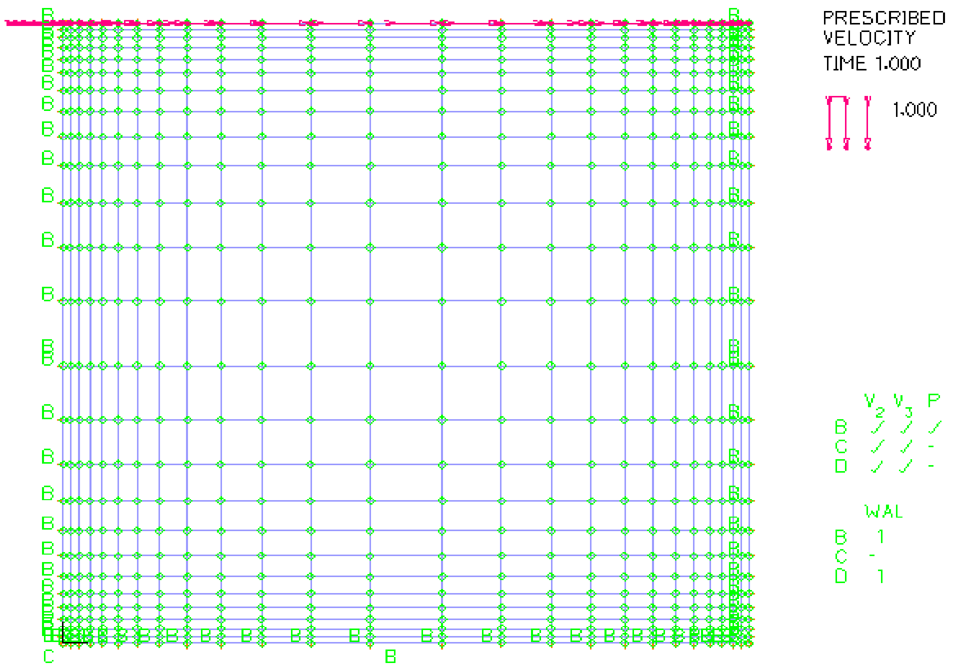
Geometry



Cavity with sliding wall

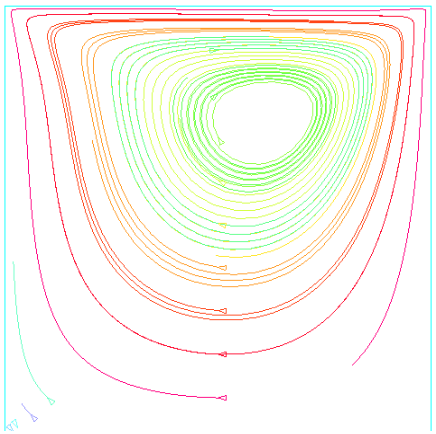
Boundary conditions

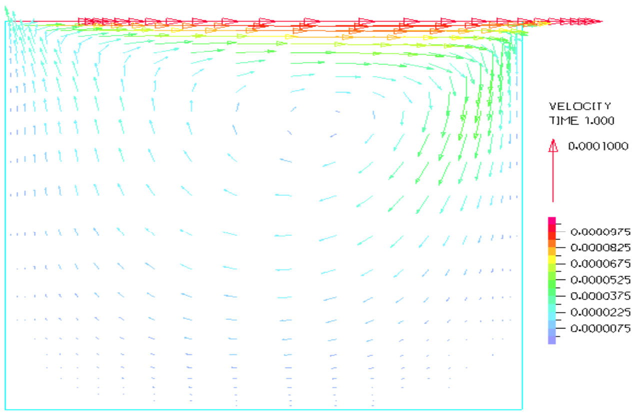
- Velocities at the upper wall
- No slip at the other walls

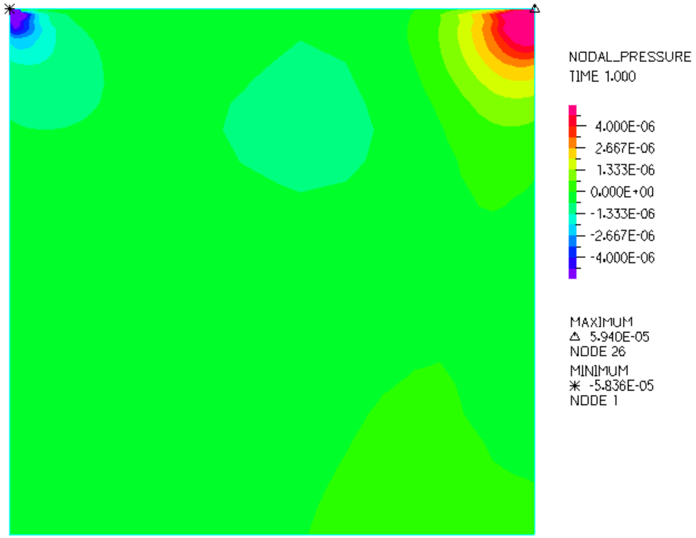


Cavity with sliding wall

Laminar flow





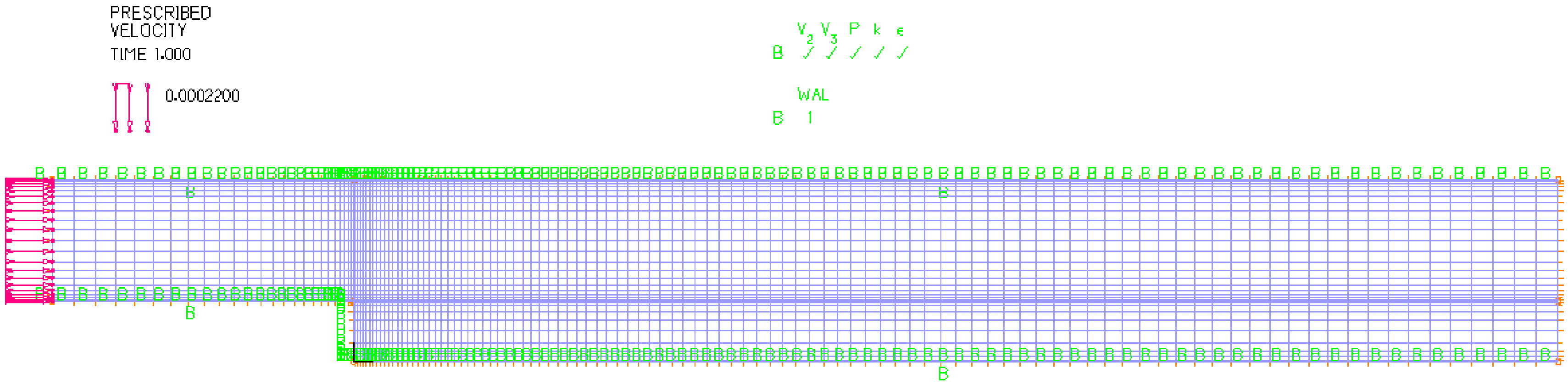


Backward Facing Step

Geometry

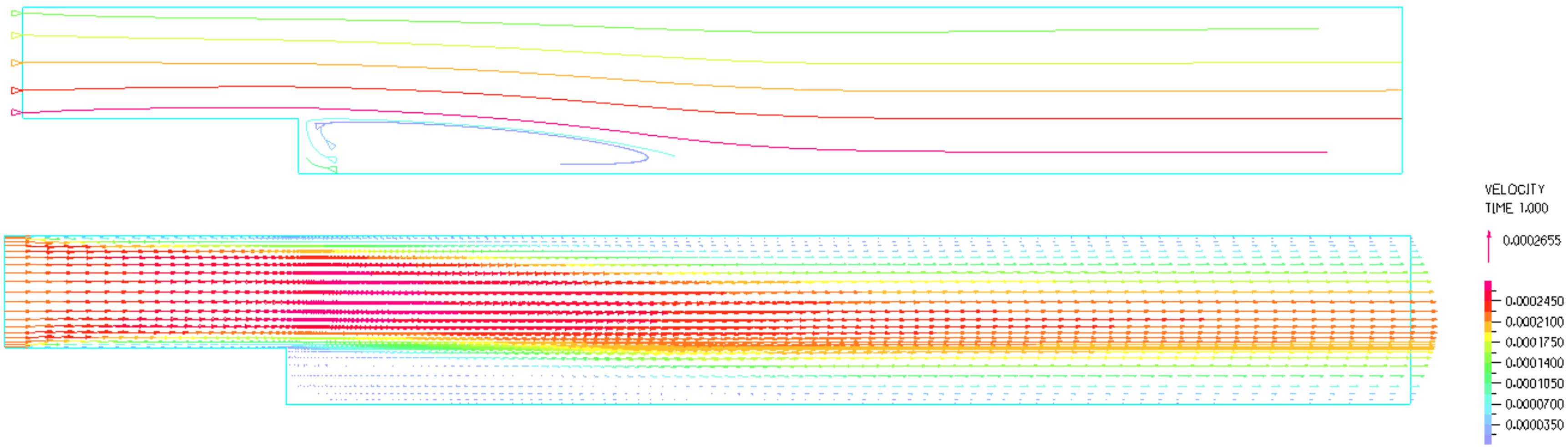
Boundary conditions at the walls:

- No slip (B´s verdes)



Backward Facing Step

Laminar flow



Velocity = 0.00022 m/s
Reynolds = 440