On the convergence of incompressible finite element formulations: the Patch Test and the Inf-Sup condition

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Abstract

Engineers have developed robust and efficient incompressible finite element formulations using tools such as the Patch Test and the counting of constraints / variables, the first one aimed at the development of consistent elements and the second one aimed at the development of non-locking and stable elements. The mentioned tools are rooted in the physics of the continuum mechanics problem. Mathematicians, on the other side, developed complex and powerful tools to examine the convergence of finite element formulations, such as the inf-sup condition, these methods are based on the properties of the elliptical PDEs that constitute the mathematical model of the continuum mechanics problem.

In this paper we intend to understand the inf-sup condition from an engineering perspective, so as to be able to incorporate it to the package of tools used in the development of finite element formulations.

1 Introduction

Problems such as incompressible elasticity and fluid dynamics of incompressible flows have always constituted a challenge to finite element developers: the standard displacement (velocity) interpolated elements usually suffer from *locking* [2] [23] and to obtain valid results mixed formulations are required . Formulations that have been successfully used to overcome the locking problem, and that we will consider in this paper, are:

- Mixed u/p formulations in which the displacements (velocities) are interpolated with C^0 continuity, while the pressure is represented using an interpolation discontinuous between elements [14] [23].

- Mixed penalty formulations in which the displacements (velocities) are interpolated with C^0 continuity, while the volumetric strain (volumetric strain rate) is represented using an interpolation discontinuous between elements; the results provided by these elements tend to the results of the equivalent u/p elements when the penalty coefficient tends to infinity [14] [23].

- Mixed augmented Lagrangian formulations in which the displacements (velocities) are interpolated with C^0 continuity, while the volumetric strain (volumetric strain rate) is represented using an interpolation discontinuous between element; the results provided by these elements tend to the results of the equivalent u/p elements when the number of iterations in the augmentation procedure tends to infinity. The advantage of the augmented Lagrangian technique over the standard penalty method is the possibility of using a smaller penalty parameter and therefore better conditioned matrices [9].

Using the above formulations "finite element designers" have developed elements that avoid the locking problem and converge to the displacements (velocities) that result from the *exact* solution of the problem (*consistency*). However, in some cases, the pressure predictions of those elements may be polluted with non-physical *pressure modes* [14] [18] [19], that is to say an unstable pressure prediction may be encountered (*stability*).

New u/p formulations with the displacements (velocities) and the pressure interpolated with C^0 continuous functions have been lately developed and they are free from pressure modes [15].

To investigate the consistency of a formulation we use the standard Patch Test together with ad hoc Patch Tests designed to specifically test incompressible formulations: highly $u-$constrained problems in which, to analyze the formulation potential for locking, a balance is established between the incompressibility constraints and the available $u-$degrees of freedom. Also a balance between the $u-$degrees of freedom and the $p-$degrees of freedom provides a first indication of the formulation stability (necessary but not sufficient stability condition) [14] [20] [22] [23] [24]. In the second section of this paper we discuss the Patch Test.

To investigate the possibility of pressure modes (*checkerboard pressure predictions*) there is not a simple numerical tool such as the Patch Test and the finite element developer usually resigns an a priori analysis, codes the element and performs abundant numerical experimentation. Even tough the inf-sup condition [1] [4] is an available analytical tool to investigate the formulation potential for unstable pressure predictions, its use as a finite element development tool is quite cumbersome. Only lately a numerical test based on the inf-sup condition was presented, but its use is still not a very simple matter [5] [7]. In the third section of this paper we discuss a *continuum mechanics interpretation* of the inf-sup condition and use a discrete version of it [13] to analyze several element formulations.

2 The Patch Test

Irons' Patch Test is a blend of sound engineering intuition and profound knowledge on the behavior of finite element approximations to continuum mechanics mathematical models. We shall try to understand the motivation of the Patch Test examining a very simple example.

In a standard elasticity or fluid mechanics problem let us call $\widehat{u}_i(x_1, x_2, x_3)$ for $(i = 1, 2, 3)$ the displacement or velocity Cartesian components that, in the neighborhood of a point P, exactly satisfy the mathematical model.

Let us call u_i^h the finite element approximation to $\widehat{u}_i$ obtained with a mesh characterized by an element dimension, h. The finite element solution at other point different from P can be written as:

$$u_i^h = [u_i]_P^h + \left[\frac{\partial u_i}{\partial x_j}\right]_P^h \Delta x_j + \frac{1}{2}\left[\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right]_P^h \Delta x_j \Delta x_k + \ ... \tag{1}$$

if the value of $[u_i]_P^h$ coincides with the exact solution at P, then

$$u_i^h \underset{\Delta x_j \to 0}{\rightarrow} \widehat{u}_i \tag{2a}$$

$$\textit{if and only if} \quad \left[\frac{\partial u_i}{\partial x_j}\right]_P^h = \left[\frac{\partial \widehat{u}_i}{\partial x_j}\right]_P \quad . \tag{2b}$$

Remark 1 *In order to be able to converge to the exact continuum solution when the mesh is refined ($h \to 0$) the finite element formulation has to be capable of representing exactly the derivatives $\left[\frac{\partial \widehat{u}_i}{\partial x_j}\right]_P$ for any mesh size.*

The Patch Test was designed to assess on the fulfillment of the above condition. In Fig. 1 we represent a typical mesh used for the Patch Test when examining quadrilateral plane elements. The patch of elements is subjected to nodal point displacement (velocity) constraints just sufficient to remove all physical rigid body modes, and is subjected to externally applied boundary nodal point forces that correspond to constant boundary stress conditions. The analysis yields the nodal point displacements (velocities) and the internal element stresses. The Patch test is passed if this predicted quantities exactly match the analytical solution.

The fulfillment of the Patch Test indicates that the necessary condition for convergence is achieved, although the actual convergence may be very slow. To identify what order of stress variation can be reached, within acceptable convergence rates, higher-order Patch Test need to be performed [3].

3 The inf - sup condition

In this section we will derive the inf-sup condition based on the physics of the continuum mechanics problem we are trying to solve; for these purposes we will consider a linear problem in which we seek for the displacement or velocity field $\underline{u}$ that satisfies the Principle of Virtual Work (equilibrium) [2]:

$$\int_V \widehat{\underline{\underline{\sigma}}} : \delta\underline{\underline{\varepsilon}}\ dv = \int_{V(S)} \underline{f} \cdot \delta\underline{u}\ dv(ds)\ . \tag{3}$$

In the above equation, $\widehat{\underline{\underline{\sigma}}}$ is the stress tensor that satisfies equilibrium; $\underline{\underline{\varepsilon}}$ is the infinitesimal strain tensor in the case of elasticity problems or strain rate tensor in the case of fluid flows; V is the continuum body volume and S its external surface. The external loads per unit volume or unit external surface are $\underline{f}$ while the continuum body displacements or velocities are $\underline{u}$.

For the discretized problem we have,

$$\int_V \underline{\underline{\sigma}}^h : \delta\underline{\underline{\varepsilon}}^h\ dv = \int_{V(S)} \underline{f} \cdot \delta\underline{u}^h\ dv(ds) \tag{4}$$

where $(.)^h$ indicates the magnitudes corresponding to the discretized problem.

In what follows we analyze a number of mathematical properties that are fulfilled by the discretized quantities (see [2] for a more rigorous presentation of these properties).

3.1 Bounds for the stress error

The *displacements error* for an $h-$discretization is,

$$\underline{e}^h = \widehat{\underline{u}} - \underline{u}^h\ . \tag{5}$$

Equation (3) corresponding to the continuum is satisfied for any pair $(\delta\underline{u}\ , \delta\underline{\underline{\varepsilon}})$ that satisfies:

- The rigid (essential) boundary conditions.
- The compatibility conditions [17].

In particular, Eqn. (3) is satisfied for the pair $(\delta\underline{u}^h,\ \delta\underline{\underline{\varepsilon}}^h)$; therefore we can write,

$$\int_V \widehat{\underline{\underline{\sigma}}} : \delta\underline{\underline{\varepsilon}}^h\ dv = \int_{V(S)} \underline{f} \cdot \delta\underline{u}^h\ dv(ds) \tag{6}$$

using Eqns. (4) and (6) we get,

$$\int_V (\widehat{\underline{\underline{\sigma}}} - \underline{\underline{\sigma}}^h) : \delta\underline{\underline{\varepsilon}}^h\ dv = 0\ . \tag{7}$$

Remark 2 *The above equation only states the well known fact that in a finite element model, the nodal loads equivalent "in the virtual work sense" to the external loads, are exactly equilibrated by the nodal loads, equivalent in the "in the virtual work sense" to the stresses.*

3.2 Bounds for the energy error

For a linear hyperelastic material under a conservative loading we can define the Potential Energy (Π), in this case the Principle of Virtual Energy leads to [21],

$$\delta\Pi = 0 \tag{8}$$

where,

$$\Pi = U - V \tag{9a}$$

$$U = \frac{1}{2}\int_V C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl}\,dv \geq 0\ . \tag{9b}$$

In the above equations, U is the elastic energy and V is the loads potential energy; C_{ijkl} are the Cartesian components of the fourth order elastic constitutive tensor and ε_{ij} are the strain components (we are considering only infinitesimal strains).

Assuming a discretized solution that fulfills the Patch Test, the error in the strain components is,

$$e^h_{ij} = \widehat{\varepsilon}_{ij} - \varepsilon^h_{ij} \tag{10}$$

hence, considering the symmetries in the constitutive tensor [17] we can write,

$$\widehat{U} = \frac{1}{2}\int_V C_{ijkl}\,(\varepsilon^h_{ij}\,\varepsilon^h_{kl} + e^h_{ij}\,e^h_{kl} + 2\,\varepsilon^h_{ij}\,e^h_{kl})\,dv \geq 0 \tag{11a}$$

$$\widehat{U} = U_h + \frac{1}{2}\int_V C_{ijkl}\,e^h_{ij}\,e^h_{kl}\,dv + \int_V C_{ijkl}\,\varepsilon^h_{ij}\,e^h_{kl}\,dv \geq 0\ . \tag{11b}$$

For the last integral in the above equation we can write,

$$\int_V C_{ijkl}\,\varepsilon^h_{ij}\,e^h_{kl}\,dv = \int_V (\underline{\underline{\widehat{\sigma}}} - \underline{\underline{\sigma}}^h) : \underline{\underline{\varepsilon}}^h\,dv \tag{12}$$

since in Eqn. (7), $\underline{\underline{\varepsilon}}^h$ fulfills all the requirements to be a valid $\delta\underline{\underline{\varepsilon}}^h$, the above integral is equal to zero. Considering that $\int_V C_{ijkl}\,e^h_{ij}\,e^h_{kl}\,dv \geq 0$ we get,

$$\widehat{U} \geq U_h\ . \tag{13}$$

Remark 3 *The exact value of the elastic energy is always bigger than the value of the elastic energy provided by the finite element model, except when the finite element model solution is coincident with the exact solution.*

3.3 Energy of the strain error field

Let us consider in the continuum problem the following strain field: $[e^h_{ij} + \delta\varepsilon^h_{ij}]$; the elastic energy associated to it is,

$$U(e^h_{ij} + \delta\varepsilon^h_{ij}) = \frac{1}{2}\int_V C_{ijkl}\, e^h_{ij}\, e^h_{kl}\, dv + \int_V C_{ijkl}\, e^h_{ij}\, \delta\varepsilon^h_{kl}\, dv + \frac{1}{2}\int_V C_{ijkl}\, \delta\varepsilon^h_{ij}\, \delta\varepsilon^h_{kl}\, dv\ . \quad (14)$$

The second integral on the l.h.s. of the above equation is zero, due to the result in Eqn. (7); therefore,

$$U(e^h_{ij} + \delta\varepsilon^h_{ij}) = U(e^h_{ij}) + U(\delta\varepsilon^h_{ij}) \Longrightarrow U(e^h_{ij}) \leq U(e^h_{ij} + \delta\varepsilon^h_{ij}) \quad (15)$$

the above inequality can also be written in the following ways,

$$U(\underline{\underline{\widehat{\varepsilon}}} - \underline{\underline{\varepsilon}}^h) \leq U(\underline{\underline{\widehat{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta\underline{\underline{\varepsilon}}^h) \quad (16a)$$

$$\int_V (\underline{\underline{\widehat{\sigma}}} - \underline{\underline{\sigma}}^h) : (\underline{\underline{\widehat{\varepsilon}}} - \underline{\underline{\varepsilon}}^h)\, dv \leq \int_V (\underline{\underline{\widehat{\sigma}}} - \underline{\underline{\sigma}}^h + \delta\underline{\underline{\sigma}}^h) : (\underline{\underline{\widehat{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta\underline{\underline{\varepsilon}}^h)\, dv\ . \quad (16b)$$

Remark 4 *The elastic energy associated to the strain error field is minimum.*

3.4 Upper bound for the strain energy

In a well behaved problem it must be possible, even for the compressibility modulus $\kappa \longrightarrow \infty$, to define a finite number $M > 0$ such that,

$$U_h \leq M \int_V \underline{\underline{\varepsilon}}^h : \underline{\underline{\varepsilon}}^h\, dv \quad (17)$$

where M depends on the actual elasticity problem being considered and on its material constants, but not on the strain field $\underline{\underline{\varepsilon}}^h$.

A finite element formulation may not fulfill the above condition if for example:

1. Being the pressure and strain fields non-continuous between elements, the inter-element energy is unbounded.
2. The element formulation locks (e.g. isoparametric plate / shell elements and standard isoparametric displacement based elements in incompressibility problems [2])

Of course, for any strain field in the continuum problem, the above condition must be fulfilled, for example,

$$U(\underline{\underline{\widehat{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta\underline{\underline{\varepsilon}}^h) \leq M \int_V (\underline{\underline{\widehat{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta\underline{\underline{\varepsilon}}^h) : (\underline{\underline{\widehat{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta\underline{\underline{\varepsilon}}^h)\, dv\ . \quad (18)$$

3.5 Lower bound for the strain energy

In order to be free of "spurious zero energy modes" it must always be possible to define a number $\alpha > 0$ such that,

$$U_h \geq \alpha \int_V \underline{\underline{\varepsilon}}^h : \underline{\underline{\varepsilon}}^h \, dv \tag{19}$$

where α depends on the actual elasticity problem being considered and on its material constants, but not on the strain field $\underline{\underline{\varepsilon}}^h$.

Of course, for any strain field in the continuum problem, the above condition must be fulfilled, for example,

$$U(\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h) \geq \alpha \int_V (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h) : (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h) \, dv \quad . \tag{20}$$

3.6 Material indifference

In Eqns. (17) and (19) the constants M and α depend on the material parameters. In this subsection we discuss the requirements for these constants to be material indifferent.

Using Eqns. (20), (16a) and (18) we can write,

$$\alpha \int_V \left(\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h\right) : \left(\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h\right) \, dv \leq \inf_{\forall \delta \underline{\underline{\varepsilon}}^h} U(\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta \underline{\underline{\varepsilon}}^h) \leq M \inf_{\forall \delta \underline{\underline{\varepsilon}}^h} \int_V (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta \underline{\underline{\varepsilon}}^h) : (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta \underline{\underline{\varepsilon}}^h) \, dv \tag{21}$$

and therefore,

$$\int_V \left(\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h\right) : \left(\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h\right) \, dv \leq \frac{M}{\alpha} \inf_{\forall \delta \underline{\underline{\varepsilon}}_h} \int_V (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta \underline{\underline{\varepsilon}}^h) : (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta \underline{\underline{\varepsilon}}^h) \, dv \; . \tag{22}$$

We define,

$$d_E = \inf_{\forall \delta \underline{\underline{\varepsilon}}_h} \int_V (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta \underline{\underline{\varepsilon}}^h) : (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta \underline{\underline{\varepsilon}}^h) \, dv \tag{23}$$

the *energy distance* from the exact solution to the solutions contained in the space of the possible finite element solutions. Convergence of the finite element solution means,

$$\lim_{h \longrightarrow 0} \inf_{\forall \delta \underline{\underline{\varepsilon}}_h} \int_V (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta \underline{\underline{\varepsilon}}^h) : (\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h + \delta \underline{\underline{\varepsilon}}^h) \, dv = 0 \; . \tag{24}$$

Since, as we stated above, our objective is to have the same convergence rate for any set of material constants, we need:

$$\int_V \left(\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h\right) : \left(\widehat{\underline{\underline{\varepsilon}}} - \underline{\underline{\varepsilon}}^h\right) \, dv \leq c \, d_E \tag{25}$$

where c is a constant independent of the mesh size (h) and of the material constants.

3.7 Application to incompressible materials

We can decompose the stress and strain tensor into their deviatoric and hydrostatic components; therefore,

$$\underline{\underline{\sigma}} = \underline{\underline{s}} + p\,\underline{\underline{g}} \tag{26a}$$

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}' + \frac{1}{3}\,\varepsilon_v \underline{\underline{g}} \tag{26b}$$

where, $\underline{\underline{\sigma}}$ is the stress tensor, $\underline{\underline{s}}$ its deviatoric part, $\underline{\underline{g}}$ the spatial metric tensor and p the hydrostatic stress component; $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\varepsilon}}'$ are the strain tensor and deviatoric strain tensor respectively when analyzing a solid and the strain rate and deviatoric strain rate tensors when analyzing a fluid, while ε_v is the volumetric strain (strain rate) component.

The constitutive law for an incompressible linear elastic (or Newtonian fluid) material is,

$$\underline{\underline{s}} = 2\,\mu\,\underline{\underline{\varepsilon}}' \tag{27a}$$

$$p = \kappa\,\varepsilon_v \tag{27b}$$

in the above $\kappa \longrightarrow \infty$; $\varepsilon_v \longrightarrow 0$ and p remains a finite number.

For the continuum and discretized problems we can write,

$$U = U_D + U_V = \mu \int_V \underline{\underline{\varepsilon}}' : \underline{\underline{\varepsilon}}'\, dv + \frac{\kappa}{2} \int_V (\varepsilon_v)^2\, dv \tag{28a}$$

$$U^h = U_D^h + U_V^h = \mu \int_V \underline{\underline{\varepsilon_h}}' : \underline{\underline{\varepsilon_h}}'\, dv + \frac{\kappa}{2} \int_V \left({\varepsilon_v}^h\right)^2 dv \quad . \tag{28b}$$

For this case, and taking into consideration that $\widehat{\varepsilon}_v = 0$, Eqn. (7) can be written as,

$$\int_V \left(\widehat{\underline{\underline{s}}} - \underline{\underline{s}}^h\right) : \delta\underline{\underline{\varepsilon}}_h'\, dv + \int_V \left(\widehat{p} - p^h\right)\, \delta{\varepsilon_v}^h\, dv = 0 \tag{29a}$$

$$\int_V \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h'\right) : \delta\underline{\underline{\varepsilon}}_h'\, dv + \frac{\kappa}{2\mu} \int_V \varepsilon_v^h\, \delta{\varepsilon_v}^h\, dv = 0 \tag{29b}$$

and Eqn. (13) as,

$$\int_V \widehat{\underline{\underline{\varepsilon}}}' : \widehat{\underline{\underline{\varepsilon}}}'\, dv \geq \int_V \underline{\underline{\varepsilon}}_h' : \underline{\underline{\varepsilon}}_h'\, dv + \frac{\kappa}{2\mu} \int_V \left({\varepsilon_v}^h\right)^2\, dv\ . \tag{30}$$

Also, for the incompressible materials we can write the property in Eqns. (16a) and (16b) as,

$$\int_V \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h'\right) : \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h'\right)\, dv + \frac{\kappa}{2\mu}\int_V \left({\varepsilon_v}^h\right)^2\, dv \leq$$
$$\int_V \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h' + \delta\underline{\underline{\varepsilon}}_h'\right) : \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h' + \delta\underline{\underline{\varepsilon}}_h'\right)\, dv + \frac{\kappa}{2\mu}\int_V \left({\varepsilon_v}^h + \delta{\varepsilon_v}^h\right)^2\, dv\ .$$

In order to be free of the locking phenomena it must be possible, in the incompressible case ($\kappa \longrightarrow \infty$), to define a finite number $M > 0$ such that,

$$U_h \leq M\left[\frac{1}{\kappa}\int_V \underline{\underline{\varepsilon}}_h' : \underline{\underline{\varepsilon}}_h'\, dv + \frac{1}{2\mu}\int_V \left({\varepsilon_v}^h\right)^2\, dv\right]\ . \tag{31}$$

In order to be free of "spurious zero energy modes" it must be possible, in the incompressible case ($\kappa \longrightarrow \infty$), to define a number $\alpha > 0$ such that,

$$U_h \geq \alpha\left[\frac{1}{\kappa}\int_V \underline{\underline{\varepsilon}}_h' : \underline{\underline{\varepsilon}}_h'\, dv + \frac{1}{2\mu}\int_V \left({\varepsilon_v}^h\right)^2\, dv\right]\ . \tag{32}$$

If we want to have a *material indifferent convergence rate*, using Eqn. (25) we get,

$$\frac{1}{\kappa}\int_V \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h'\right) : \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h'\right)\, dv + \frac{1}{2\mu}\int_V \left({\varepsilon_v}^h\right)^2\, dv \leq$$
$$c\left(\frac{1}{\kappa}\int_V \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h' + \delta\underline{\underline{\varepsilon}}_h'\right) : \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h' + \delta\underline{\underline{\varepsilon}}_h'\right)\, dv + \frac{1}{2\mu}\int_V \left({\varepsilon_v}^h + \delta{\varepsilon_v}^h\right)^2\, dv\right)\quad .$$

Calling $\beta = 1/c$ we get that for all possible values of $\delta\underline{\underline{\varepsilon}}_h$ the finite element formulation has to fulfill the following inequality

$$\frac{\frac{1}{\kappa}\int_V \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h' + \delta\underline{\underline{\varepsilon}}_h'\right) : \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h' + \delta\underline{\underline{\varepsilon}}_h'\right)\, dv + \frac{1}{2\mu}\int_V \left({\varepsilon_v}^h + \delta{\varepsilon_v}^h\right)^2\, dv}{\frac{1}{\kappa}\int_V \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h'\right) : \left(\widehat{\underline{\underline{\varepsilon}}}' - \underline{\underline{\varepsilon}}_h'\right)\, dv + \frac{1}{2\mu}\int_V \left({\varepsilon_v}^h\right)^2\, dv} \geq \beta > 0\ . \tag{33}$$

When $\kappa \rightarrow \infty$, since $p^h = \kappa{\varepsilon_v}^h$ is a finite number, we get,

$$\frac{\int_V (p^h + \delta p^h)\ \left({\varepsilon_v}^h + \delta{\varepsilon_v}^h\right)\, dv}{\int_V p^h\ {\varepsilon_v}^h\, dv} \geq \beta > 0\ . \tag{34}$$

If the maximum reached by the above equation, considering the different possible values of $\delta{\varepsilon_v}^h$, is zero then it means that the denominator tends to infinity when $\kappa \longrightarrow \infty$ and therefore the element formulation locks; that is to say, the element interpolations are not capable of representing an incompressible situation.

On the other hand, if the minimum value reached by the above equation, considering the different possible values of δp^h, is zero then it means that the finite element formulation incorporates "spurious rigid body modes".

Hence, using the standard notation we can write [2],

$$\inf_{\forall \delta p^h} \sup_{\forall \delta {\varepsilon_v}^h} \frac{\int_V (p^h + \delta p^h) \ \left({\varepsilon_v}^h + \delta {\varepsilon_v}^h\right) \ dv}{\int_V p^h \ {\varepsilon_v}^h \ dv} \geq \beta > 0 \tag{35}$$

For a formulation that can interpolate "spurious rigid body modes" $\int_V (p^h + \delta p^h) \ \left({\varepsilon_v}^h + \delta {\varepsilon_v}^h\right) \ dv = 0$.

In a finite element discretization in which,

$$\underline{u}^h = \underline{H}_u \underline{U} \tag{36}$$

$$\underline{\varepsilon}'_h = \underline{B}_D \underline{U} \tag{37}$$

$$\varepsilon_v^h = \underline{B}_v \underline{U} \tag{38}$$

$$p^h = \underline{H}_p \underline{P} \tag{39}$$

and,

$\underline{U}$: vector of nodal displacements or velocities,

$\underline{P}$: vector of element pressures,

we can write [24] for an u/p and penalty formulations,

$$\begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{up}^T & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{U} \\ \underline{P} \end{bmatrix} = \begin{bmatrix} \underline{R} \\ \underline{0} \end{bmatrix} \tag{40a}$$

$$\begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{up}^T & -\frac{1}{\kappa}\underline{I} \end{bmatrix} \begin{bmatrix} \underline{U} \\ \underline{P} \end{bmatrix} = \begin{bmatrix} \underline{R} \\ \underline{0} \end{bmatrix} \tag{40b}$$

where,

$$\underline{K}_{uu} = \int_V \underline{B}_D^T \ 2\mu \ \underline{B}_D \ dv \tag{41}$$

$$\underline{K}_{up} = \int_V \underline{B}_v^T \underline{H}_p \ dv \tag{42}$$

the non-fulfillment of Eqn. (35) implies that there exists a vector of element pressures, $\underline{P}_{pm}$, that satisfies the equation,

$$\underline{K}_{up} \ \underline{P}_{pm} = \underline{0} \ . \tag{43}$$

These vectors $\underline{P}_{pm}$ are called *pressure modes* [18] [19].

3.8 Stability of the pressure prediction

As it was shown in Refs. [18] and [19] for those element formulations for which pressure modes exist, some models may provide non-unique pressure predictions while always providing unique velocity predictions. Actual numerical analyses produce a linear combination of the physical pressure plus the pressure modes, usually in the form of a checkerboard. Many a posteriori techniques have been developed to filter out from the model pressure predictions the checkermodes [14]; also in Ref. [6] an a priori technique is presented to avoid the contamination of the pressure prediction with checkermodes.

4 Analysis of finite element formulations

To illustrate with a couple of simple applications the discussion presented in the previous section we will analyze some simple finite element formulations that satisfy the Patch Test.

In all cases we will consider the simple mesh shown in Fig. 2, in which all the nodes on the external perimeter are prevented from moving.

4.1 The standard Q1-P0 element

This is one of the most used element formulations; having a bilinear velocity interpolation and a constant pressure prediction, discontinuous between elements. The number of velocity degrees of freedom is $n_u = 2$ and the number of pressure degrees of freedom is $n_p = 4$. In this case,

$$\underline{K}_{up} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \quad . \tag{44}$$

The two pressure modes that satisfy Eqn. (43) are:

$$\underline{P}_{pm}^{(H)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{45}$$

$$\underline{P}_{pm}^{(CB)} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad . \tag{46}$$

While the pressure mode in Eqn. (45) represents a hydrostatic pressure distribution the pressure mode in Eqn. (46) represents a checkerboard pressure distribution. The solution provided by the numerical analysis is a combination of the expected physical solution plus the hydrostatic mode plus the checkerboard mode.

In a well posed problem, at least one of the nodes on the domain surface must have a natural boundary condition, therefore the hydrostatic pressure mode is prevented; however, the checkerboard mode has to be filtered out.

4.2 The Q1-P0 element plus a fifth displacement (velocity) node

In this case we enrich the Q1-P0 element formulation with a central displacement (velocity) node; hence, for the problem in Fig. 2 the number of velocity degrees of freedom is $n_u = 10$ and the number of pressure degrees of freedom is $n_p = 4$. For this element,

$$\underline{K}_{up} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{47}$$

It is obvious that the introduction of a fifth displacement (velocity) node does not introduce any change with respect to the fulfillment of the inf-sup condition and the pressure modes are still the ones in Eqns. (45) and (46).

4.3 The Q1-P0 element plus a fifth displacement (velocity) node with an enriched pressure field

In this case we keep the displacement (velocity) interpolation of the previous example but we enrich the pressure interpolation using inside each element an interpolation of the form,

$$p = p_o + p_r \; r + p_s \; s \; . \tag{48}$$

For this element,

$$
\underline{K}_{up} = \begin{bmatrix}
1 & \frac{4}{9} & \frac{1}{3} & -1 & \frac{4}{9} & -\frac{1}{3} & -1 & \frac{4}{9} & \frac{1}{3} & 1 & \frac{4}{9} & -\frac{1}{3} \\
1 & \frac{1}{3} & \frac{4}{9} & 1 & -\frac{1}{3} & \frac{4}{9} & -1 & \frac{1}{3} & \frac{4}{9} & -1 & -\frac{1}{3} & \frac{4}{9} \\
0 & -\frac{16}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{16}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{16}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{16}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{16}{9} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{16}{9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{16}{9} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} . \tag{49}
$$

The two pressure modes that satisfy Eqn. (43) are:

$$
\underline{P}_{pm}^{(H)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{50}
$$

$$
\underline{P}_{pm}^{(CB)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} . \tag{51}
$$

It is obvious that the introduction of a fifth displacement (velocity) node plus an enriched pressure interpolation does not introduce any change with respect to the fulfillment of the inf-sup condition and the pressure modes are still a hydrostatic pressure mode (Eqn. (50)) and a checkerboard pressure mode (Eqn. (51)).

Remark 5 *In the above three examples we analyzed finite element formulations that while fulfilling the Patch Test do not satisfy the inf-sup condition. The three element formulations when used in numerical modeling of incompressible situations provide stable displacement (velocity) predictions but the pressure predictions are polluted with the pressure modes.*

5 Conclusions

A finite element formulation that fulfills the constant stress Patch Test will converge to the *exact displacements (velocities) continuum solution* under any geometrical or loading conditions and considering any material parameters. Even if in some case the convergence may be so slow that the element is useless for practical applications, we can assure that if we keep refining the mesh ($h \to 0$) the exact displacements (velocities) continuum solution will be reached. If the constant stress Patch Test is not fulfilled the convergence to the exact displacements (velocities) continuum solution under every circumstances cannot be assured; therefore, finite element formulations that do not fulfill the constant stress Patch Test have to be disregarded for actual engineering applications.

The fulfillment of higher order Patch Test will guarantee not only convergence when $h \to 0$ but also a reasonable convergence velocity.

Even tough the Patch Test is a tool developed for analyzing linear element formulations, it has proven to be an extremely valuable tool for analyzing non-linear element formulations [8], [11].

Regarding our topic, the finite element analysis of incompressible problems, we may state as a first conclusion that we can only consider as acceptable finite element formulations the ones that fulfill the constant stress Patch Test and in the higher order Patch Tests and/or benchmark problems present an acceptable convergence velocity.

The elements that fulfill the inf-sup condition will also guarantee a well behaved pressure solution. Even tough this is a very important aspect of the finite element formulation performance, it may be waived and it is actually waived in many practical applications: due to its simplicity, reliability and good performance in the prediction of the displacement (velocity) field, the two dimensional Q1-P0 element or its three dimensional counterpart, the H1-P0 element are a favorite choice in engineering applications (and a very reasonable choice too !).

In Fig. 3 we present the band plot of the pressure prediction obtained using a plane strain rigid-viscoplastic model of a metal rolling process [10] [12]. The model was developed using a quadrilateral element that while fulfilling the Patch Test does not fulfill the inf-sup condition [9]: it can be recognized that the physical pressure solution is polluted by a checkerboard-type solution. In Fig. 4 we plot the plate / rolls contact pressure distribution along the contact arc, obtained from the above band plot; obviously the checkerboard solution pollutes this pressure distribution rendering it useless for practical applications. However, also in Fig. 4 using the same model solution we again present the plate / rolls contact pressure distribution along the contact arc, but now we post-processed it from the nodal forces; since the nodal forces are not polluted by the checkerboard pressure solution (see Eqns. (40a) and (40b)) this time the proper friction hill type pressure distribution was obtained [16]. Therefore, in the development of metal forming models, when elements that do not fulfill the inf-sup condition are used, special care must be taken for modeling the pressure dependent blank / tools friction (e.g. when using a Coulomb friction model).

The results in Fig. 4 clearly indicate that an element formulation that fulfills the Patch Test but fails to fulfill the inf-sup condition can be reliably used if the software developer incorporates the proper filters to prevent the spurious pressure modes from polluting the solution.

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References

[1] Babuška I. (1971), "Finite element method with Lagrange multipliers", *Num. Math.*, **20**, pp. 179-192.

[2] Bathe K.-J. (1996), *Finite Element Procedures*, Prentice Hall.

[3] Bathe K.-J. and Dvorkin E.N. (1986), "A formulation of general shell elements - The use of mixed interpolation of tensorial components", *Int. J. Num. Methods in Engrg.*, **22**, pp. 697-722.

[4] Brezzi F. and Fortin M. (1991), *Mixed and Hybrid Finite Elements*, Springer - Verlag.

[5] Chapelle D. and Bathe K.-J. (1993), "The inf-sup test", *Computers & Structures*, **47**, pp. 537-545.

[6] Chen J.S., Pan C. and Chang T.Y.P. (1995), "On the control of pressure oscillation in bilinear displacement constant-pressure element", *Comput. Methods Appl. Mech. Engrg.*, **128**, pp. 137-152.

[7] Crisfield M.A. and Norris V.C. (2000), "Lessons for incompressible and near-incompressible elasticity drawn from the driven cavity flow problem", *Computers & Structures*, **75**, pp. 529-538.

[8] Dvorkin E.N. and Assanelli A.P. (1991), "2D finite elements with displacement interpolated embedded localization lines: the analysis of fracture in frictional materials", *Comput. Methods Appl. Mech. Engrg.*, **90**, pp. 829-844.

[9] Dvorkin E.N., Cavaliere M.A. and Goldschmit M.B. (1995), "A three field element via augmented Lagrangian for modelling bulk metal forming processes", *Comput. Mechs.*, **17**, pp.2-9.

[10] Dvorkin E.N., Cavaliere, M.A., Goldschmit M.B. and Amenta P.M. (1998), "On the modeling of steel product rolling processes", *Int. J. Forming Processes (ESAFORM)*, **1**, pp.211-242.

[11] Dvorkin E.N., Cuitiño A.M. and Gioia G. (1990), "Finite elements with displacement interpolated embedded localization lines insensitive to mesh size and distortions", *Int. J. Num. Methods in Engrg.*, **30**, pp. 541-564.

[12] Dvorkin E.N., Goldschmit M.B., Cavaliere M.A., Amenta P.M., Marini O. and Stroppiana,W. (1997), "2D finite element parametric studies of the flat rolling process", *J.Materials Processing Technology*, **68**, pp.99-107.

[13] Hartmann F. (1986),"The discrete Babuška - Brezzi condition", *Ingenieur-Archiv*, **56**, pp. 221-228.

[14] Hughes T.J.R. (1987), *The Finite Element Method*, Prentice-Hall.

[15] Hughes T.J.R., Franca L. and Balestra M. (1986), "A new finite element formulation for CFD: V. Circumventing the Babuška-Brezzi condition: a stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations", *Comput. Methods Appl. Mech. Engrg.*, **59**, pp. 85-99.

[16] Kobayashi S., Oh S.-I. and Altan T. (1989), *Metal Forming and the Finite Element Method,* Oxford University Press.

[17] Malvern L.E. (1969), *Introduction to the Mechanics of a Continuous Medium,* Prentice Hall.

[18] Sani R.L., Gresho P.M., Lee R.L. and Griffiths D.F. (1981), "The cause and cure (?) of the spurious pressures generated by certain FEM solutions of the incompressible Navier-Stokes equations: Part 1", *Int. J. Num. Methods in Fluids*, **1**, pp.17-43.

[19] Sani R.L., Gresho P.M., Lee R.L., Griffiths D.F. and Engelman, M. (1981), "The cause and cure (!) of the spurious pressures generated by certain FEM solutions of the incompressible Navier-Stokes equations: Part 2", *Int. J. Num. Methods in Fluids*, **1**, pp.171-204.

[20] Taylor R.L., Simo J.C., Zienkiewicz O.C. and Chan A.C.H. (1986), "The Patch Test - A condition for assessing FEM convergence", *Int. J. Num. Methods in Engrg.*, **22**, pp. 39-62.

[21] Washizu K. (1982), *Variational Methods in Elasticity & Plasticity (*3rd. Edition)*,* Pergamon Press.

[22] Zienkiewicz O.C., Qu S, Taylor R.L. and Nakazawa S. (1986), "The Patch Test for mixed formulations", *Int. J. Num. Methods in Engrg.*, **23**, pp. 1873-1883.

[23] Zienkiewicz O.C. and Taylor R.L. (1989), *The Finite Element Method* (4th. Edition), McGraw-Hill.

[24] Zienkiewicz O.C. and Taylor R.L. (1997), "The finite element Patch Test revisited. A computer test for convergence, validation and error estimates", *Comput. Methods Appl. Mech. Engrg.,* **149**, pp. 223-254.

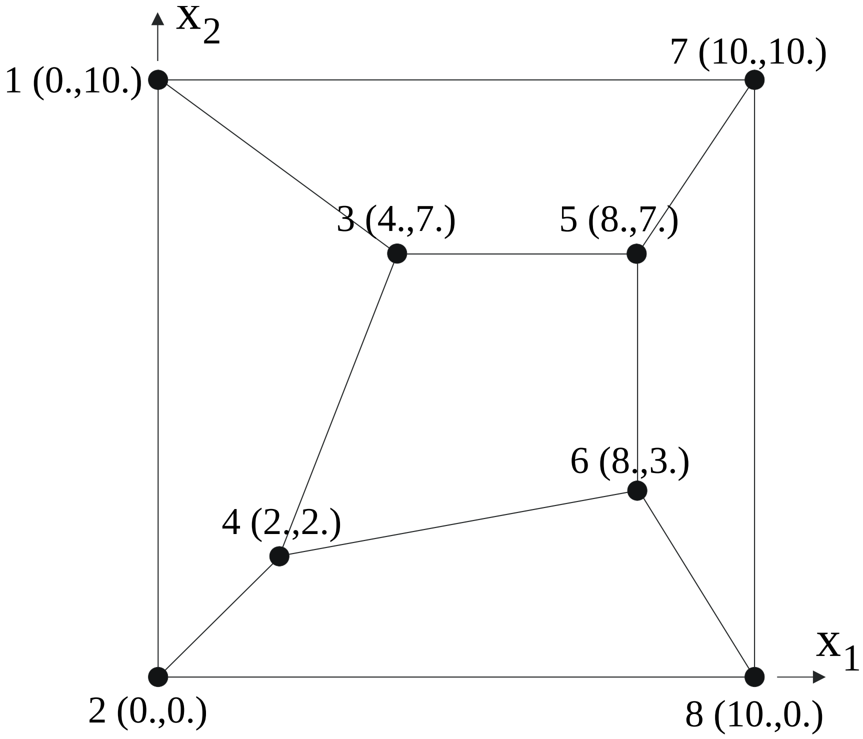

Figure 1: Patch Test mesh

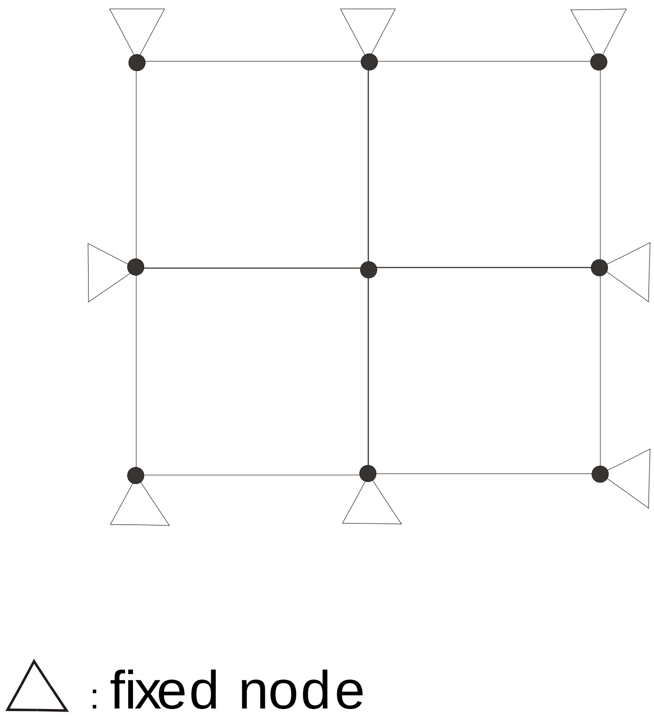

Figure 2: Four elements mesh

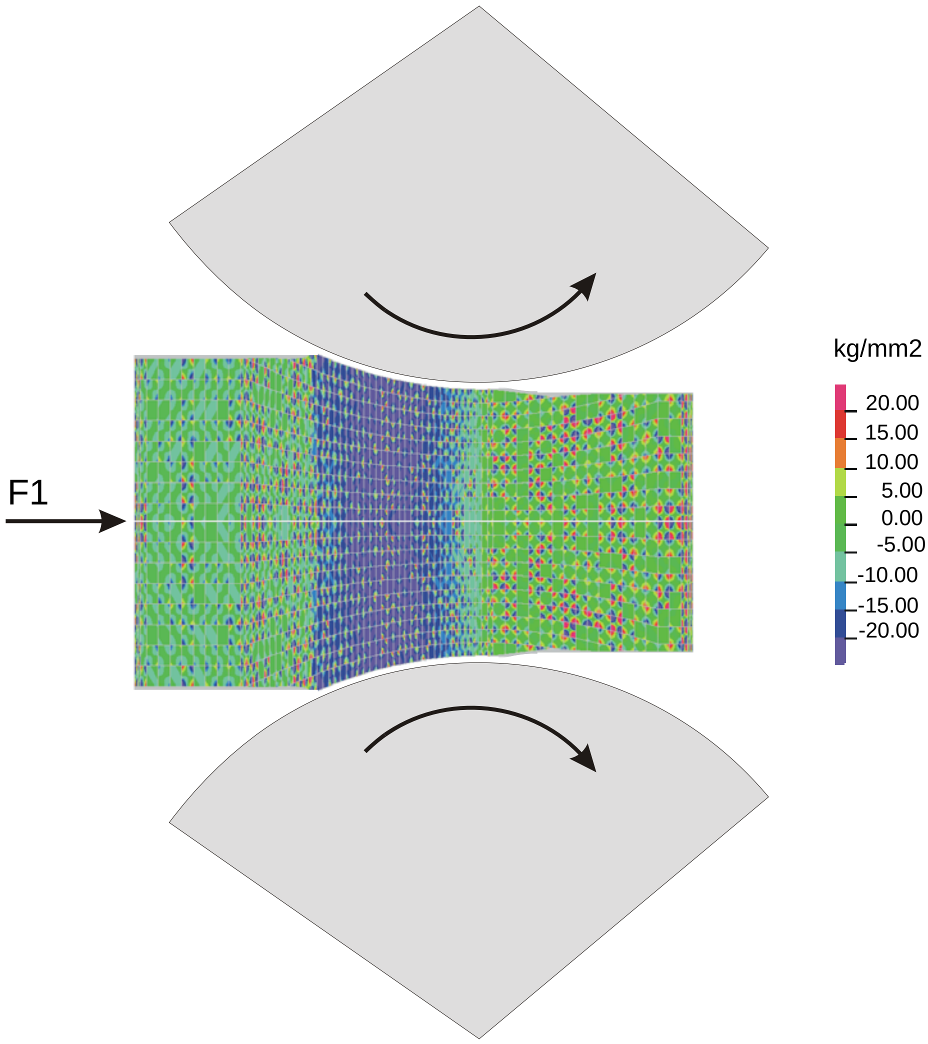

Figure 3: Modeling of plane strain rolling using the QMITC-3F element [9]. Pressure distribution

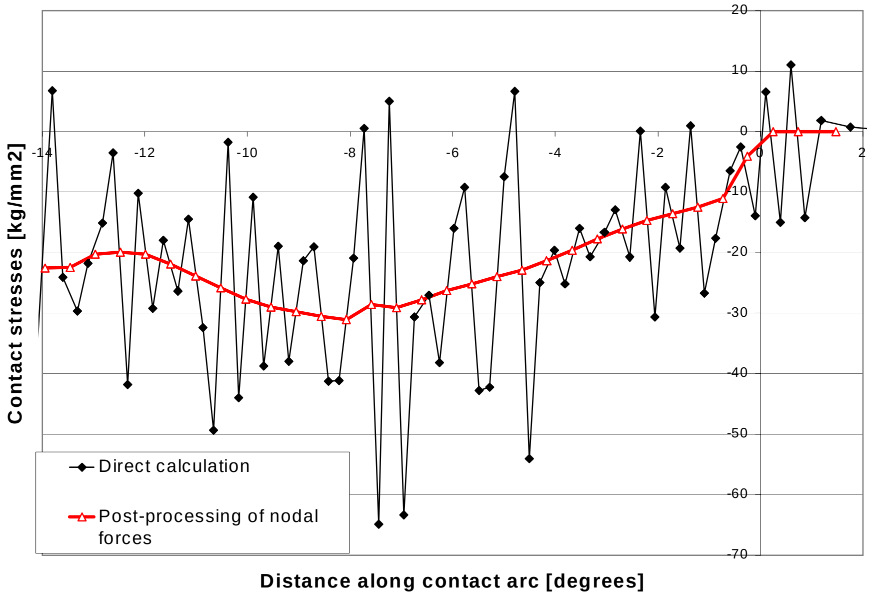

Figure 4: Modeling of plane strain rolling using the QMITC-3F element [9]. Contact pressure between rolls and plate